Well posedness and finite element approximability of time-harmonic electromagnetic boundary value problems involving bianisotropic materials and metamaterials

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Abstract

A boundary value problem for the time harmonic Maxwell system is investigated through a variational formulation which is shown to be equivalent to it and well posed if and only if the original problem is. Different bianisotropic materials and metamaterials filling subregions of the problem domain with Lipschitz continuous boundaries are allowed. Well posedness and finite element approximability of the variational problem are proved by Lax-Milgram and Strang lemmas for a class of material configurations involving bianisotropic materials and metamaterials. Belonging to this class is not necessary, yet, for well posedness and finite element approximability. Nevertheless, the material configurations of many radiation or scattering problems and many models of microwave components involving bianisotropic materials or metamaterials belong to the above class. Moreover, none of the other available tools commonly used to prove well posedness seems to be able to cope with the material configurations left out by our treatment.

Keywords: electromagnetic boundary value problems; well posedness; finite element method; convergence; bianisotropic materials.

1 Introduction

Artificial composite materials having unusual electromagnetic properties that do not occur, or are not readily available, in nature are receiving a great deal of research interest and various promising applications have been proposed for them [1]. Such “metamaterials” [1], actually consist of periodical structures, but, if we are not interested in the details of the field pattern inside them, they can be modelled by equivalent effective materials that, when isotropic, are characterized by uniformly negative real parts of effective permittivity and/or permeability [2], [3], [4]. In this case they are often called double-negative (DNG), epsilon-negative (ENG) or mu-negative (MNG) materials [5] and this classification can be readily extended to anisotropic and bianisotropic metamaterials by characterizing them by uniformly negative definite hermitian symmetric parts of effective permittivity and/or permeability [6]. Standard material are then called double-positive (DPS), of course, in order to exhaust the possible cases.

Though the basic electromagnetic phenomena occurring in metamaterials have been sometimes analyzed through simplified models having solution either analytical [5] or permitting truncated series approximation [7], the development of real applications involving metamaterials and standard media requires more detailed analysis by numerical simulation [8], [9]. On the other hand,

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in order that we can trust results of numerical simulations, they must be based on convergent approximations of well posed problem and the literature is not fully satisfactory in this respect. As a matter of fact, while many numerical simulations involving metamaterials have been reported, only few papers address well posedness of problems involving metamaterials and/or convergence of their numerical approximations. A result of well posedness and finite element approximability that cover some nontrivial configurations of anisotropic materials and metamaterials is given in [6]. In [10] finite element approximability of a wave propagation problem not involving standard media but just a homogeneous DNG medium is addressed, but well posedness of the problem is not proved. In [11], [12], [13], [14], ill-posedness of some specific problems involving simple configurations of both homogeneous and isotropic materials and metamaterials is pointed out. Well posedness of a simpler scalar model problem having some of the mathematical difficulties of a three-dimensional Maxwell problem involving an interface between two homogeneous, lossless and isotropic materials with different sign of permittivity and/or permeability is investigated in [15], as a preliminary step towards addressing the above Maxwell problem itself. The analysis of the latter is then worked out in [16], where sufficient conditions for well posedness are given, under the same hypotheses concerning the effective constitutive parameters.

The purpose of the present paper is to extend to bianisotropic materials and metamaterials the well posedness and finite element approximability result for time-harmonic electromagnetic boundary value problems involving anisotropic metamaterials that we presented in [6]. This is a worthy target because periodic structures actually implementing metamaterials sometimes exhibit a bianisotropic behaviour [17], [18]. Moreover, to the best of the authors knowledge, no well posedness result for bianisotropic materials, not even DPS, can be found in the literature. Maybe, this does not happen by accident since it seems that none of the available tools commonly used to prove well posedness does work with the Maxwell system, when different materials are present and one of them at least, is genuinely bianisotropic or biisotropic. As a matter of fact, when only isotropic and anisotropic materials are present, the problem solution is constrained by the divergence conditions to a space which is compactly embedded in an $L^2$-space and the problem fits the Fredholm alternative framework. On the other hand, when a bi(an)isotropic material is present, the available compact embeddings do not fit, in general, the different structure of the equations. They apply, for instance, if material properties are Lipschitz continuous over the whole problem domain, but this assumption rules out different materials abutting and reduces the possible covering of the model to a single bi(an)isotropic medium, a scarcely interesting case.

In [6], apart from less restrictive conditions on the two three-by-three matrix valued complex functions representing the effective dielectric permittivity and the effective magnetic permeability, the variational formulation of the problem and its equivalence to the Maxwell system were standard. On the contrary, to the best of the authors knowledge, corresponding results concerning the problem addressed in the present paper are not readily available in the literature. Therefore, we show here in full details that the Maxwell system (Section 2, Problem 1) and the variational formulation (Section 3, Problem 2) are equivalent (Section 3, Theorem 1) and that the well posedness of any of them entails the well posedness of the other one (Section 3, Theorem 2).

The same approach of [6] is then applied to the variational formulation in order to obtain conditions for well posedness and finite element approximability. In particular, it is shown that (Section 4, Theorem 5), if the main sufficient condition for well posedness found in [6] (i. e., H9) is slightly strengthened and the definitions involved in it are suitably extended to bianisotropic materials, then the so obtained condition plays the same role in the more general setting of the present paper. The crucial point is just that of extending definitions given in [6] in such a way that essentially the same proof of well posedness still works for bianisotropic (meta)materials, taking into account that, in the most general case, permittivity and permeability cannot play any more independent roles. These new definitions are given in Section 4 and their meaning is further investigated in Appendix B.

Here we can say that in [6] we defined regions where the medium was, respectively, “electric positive”, “electric negative”, “magnetic positive”, “magnetic negative”, “electric lossy” and “magnetic lossy”. Then, loosely speaking, the sufficient condition for well posedness and finite element approximability was: “the medium is either magnetic positive in the complement of the
magnetic lossy region and electric negative in the complement of the electric lossy region or the same with positive and negative exchanged”. This statement has been generalized to bianisotropic materials by the following changes. Since in the most general case a bianisotropic material cannot be passive without having losses of both types (see Appendix B), the definitions of the electric lossy and the magnetic lossy regions now imply the presence of some losses of the other type. Furthermore, in the region where the medium is neither electric lossy nor magnetic lossy it is no longer sufficient, as before, to ask that the medium itself is both magnetic positive (respectively, negative) and electric negative (respectively, positive). Hence, in that region, we have strengthened this condition by asking that the whole $6 \times 6$ matrix governing the reactive behaviour of the medium, whose off-diagonal $3 \times 3$ submatrices are responsible for the magneto-electric effects, is positive (respectively, negative).

When the sufficient conditions for well posedness are fulfilled, finite element approximability of the variational formulation holds true and can be proved arguing exactly as in [6]. In which sense finite element approximability of the Maxwell system holds true, too, as a consequence of that is pointed out.

Considerations that strictly parallel the five ones given in the introduction of [6] to circumscribe the scope of the theory proposed in it apply also in this case. In particular, since the given conditions are only sufficient, well posedness and finite element approximability may yet hold true even if they are not satisfied. However, many radiation or scattering models and many models of microwave components involving bianisotropic materials or metamaterials are covered by the here proposed theory. In fact, if bianisotropic (meta)materials are allowed in the same examples of practical applications reported in Section 7 of [6], the so obtained material configurations fit the present theory.

The paper is organized as follows. In Section 2 we define the boundary value problem for the Maxwell system we are interested in. In Section 3 the problem is recast as a variational formulation which is shown to be equivalent to it. In Section 4 sufficient conditions for well posedness of the variational formulation are found. In Section 5 it is stated that under the same conditions Galerkin and finite element approximability of the variational formulation hold true and what this fact implies about approximability of the boundary value problem for the Maxwell system is discussed. In order not to disrupt the flow of the main reasoning some topics, even important, are reported in the Appendixes. Among them, in particular, the meaning of the definitions involved in the sufficient conditions for well posedness (Appendix B) and the main proof of Section 4 (Appendix C). Finally, in Section 6 we show some practical applications of our results.

## 2 Problem definition

In this Section, we define the time-harmonic electromagnetic boundary value problem we will deal with in the rest of the paper. Both radiation, scattering and cavity problems involving different linear, local in space and time invariant, inhomogeneous bianisotropic metamaterials will be covered by our formulation. Geometrical assumptions about the boundary and the interfaces will allow edges and vertices.

Let $\Omega$ be the open, bounded and connected subset of $\mathbb{R}^3$ where the electromagnetic boundary value problem of interest will be posed. Let $\Gamma = \partial \Omega$ be its Lipschitz continuous boundary [19] (p. 4), [20] (pp. 38-39) and denote by $\mathbf{n}$ the outward unit vector normal to $\Gamma$. To shorten many statements we summarize the above hypotheses on the domain and its boundary as follows

**H1.** $\Omega \subset \mathbb{R}^3$ is open, bounded and connected,

**H2.** $\Gamma = \partial \Omega$ is Lipschitz continuous.

The constitutive relations of the most general linear, local in space and time invariant, electromagnetic media can be written as [21]

\[
\begin{cases}
D = \frac{1}{c} P \mathbf{E} + L \mathbf{B} & \text{in } \Omega \\
\mathbf{H} = M \mathbf{E} + c Q \mathbf{B} & \text{in } \Omega
\end{cases}
\]  

(1)
where \( c \) is the velocity of light in vacuum and \( L, M, P \) and \( Q \) are four 3-by-3 matrix-valued complex functions of the space point only, defined almost everywhere in \( \Omega \) and representing the effective \([22], [1]\) constitutive parameters. Other equivalent forms of the above constitutive relations are possible \([21]\) (pp. 4-9).

Very general configurations of different inhomogeneous bianisotropic metamaterials may be modelled under the following hypothesis.

**H3.** \( \Omega \) can be decomposed into \( m \) subdomains (open and connected subsets of \( \Omega \) having Lipschitz continuous boundaries) denoted \( \Omega_i, i \in I = \{1, \ldots, m\} \), \([23]\) satisfying \( \overline{\Omega} = \overline{\Omega_1} \cup \cdots \cup \overline{\Omega_m} \) (\( \overline{\Omega} \) is the closure of \( \Omega \)) and \( \Omega_i \cap \Omega_j = \emptyset \) for \( i \neq j \). The matrix-valued complex functions representing the effective constitutive parameters satisfy \([19]\) (p. 3), \([20]\) (p. 36):

\[
P|_{\Omega_k} \in (C^0(\overline{\Omega_k}))^{3 \times 3}, \quad Q|_{\Omega_k} \in (C^0(\overline{\Omega_k}))^{3 \times 3}, \quad L|_{\Omega_k} \in (C^0(\overline{\Omega_k}))^{3 \times 3}, \quad M|_{\Omega_k} \in (C^0(\overline{\Omega_k}))^{3 \times 3}, \quad \forall k \in I.
\]

Let us point out that such hypothesis is in no way restrictive for all applications of interest since the material properties need not be globally continuous \([24]\), so allowing different materials abutting.

In order to define the problem of interest we introduce the following additional notations and hypotheses. \((L^2(\Omega))^3\) is the usual Hilbert space of square integrable vector fields on \( \Omega \) with values in \( \mathbb{C}^3 \) and with scalar product given by \( (u,v)_0,\Omega = \int_\Omega v^* u \, dV \), where \( v^* \) denotes the conjugate transpose of the column vector \( v \) (we will look for \( B \) and \( D \) in this space). In order to deal with the tangential vector fields involved in the boundary condition we define \([20]\) (p. 48)

\[
L^2_\Gamma(\Gamma) = \{ v \in (L^2(\Gamma))^3 \mid v \cdot n = 0 \text{ almost everywhere on } \Gamma \},
\]
with scalar product denoted by \((u,v)_{0,\Gamma} = \int_\Gamma v^* u \, dS\). The space where we will seek \( E \) and \( H \) is \([20]\) (p. 82; see also p. 69)

\[
U = H_{L^2,\Gamma}(\text{curl},\Omega) = \{ v \in H(\text{curl},\Omega) \mid v \times n \in L^2_\Gamma(\Gamma) \},
\]
with \( H(\text{curl},\Omega) = \{ v \in (L^2(\Omega))^3 \mid \text{curl} v \in (L^2(\Omega))^3 \} \). For the precise meaning of \( v \times n \) (and \( n \times v \times n \), used later on) see Appendix A. The scalar product in the Hilbert space \( U \) is given by \([20]\) (p. 84, p. 69)

\[
(u,v)_{U,\Omega} = (u,v)_{0,\Omega} + (\text{curl} u, \text{curl} v)_{0,\Omega} + (u \times n, v \times n)_{0,\Gamma},
\]
and the induced norm is \( \| u \|_{U,\Omega} = \| (u,u)^{1/2} \|_{U,\Omega} \). The symbol \( \omega \) represents the angular frequency, which, without loss of generality for wave problems, is assumed to be real and positive. Moreover, \( J_e \) and \( J_m \) are the electric and magnetic current densities, respectively, prescribed by the sources, \( Y \) is the scalar admittance involved in impedance boundary condition and \( f_R \) is the corresponding inhomogeneous term. Finally, the admittance function \( Y \) with domain \( \Gamma \) and range in \( \mathbb{C} \) is assumed to satisfy

**H4.** \( Y \) is piecewise continuous and \(|Y| \) is bounded and nowhere vanishing.

We are now in a position to state the electromagnetic boundary value problem we will address in this paper.

**Problem 1.** Under the hypotheses H1-H4, given \( \omega > 0, J_e \in (L^2(\Omega))^3, J_m \in (L^2(\Omega))^3 \) and \( f_R \in L^2_\Gamma(\Gamma) \), find \((E,B,H,D) \in U \times (L^2(\Omega))^3 \times U \times (L^2(\Omega))^3 \) satisfying the Maxwell equations with impedance boundary conditions

\[
\begin{cases}
\text{curl} H - j\omega D = J_e & \text{in } \Omega \\
\text{curl} E + j\omega B = -J_m & \text{in } \Omega \\
H \times n - Y(n \times E \times n) = f_R & \text{on } \Gamma
\end{cases}
\]
and the constitutive relations (1).
Note that (5) with hypothesis H4 can be used to enforce lowest order absorbing boundary conditions [25] (p. 9), so that the above model can be thought of as an approximation of a radiation or scattering problem, or boundary conditions at imperfectly conducting surfaces [26] (pp. 384-385), so that the above model can be thought of as a realistic formulation of a cavity problem.

More complex boundary conditions could be considered as well. However, we chose the above simple model since, on the one hand, the generality of our results is not reduced in a significant way and, on the other hand, the mathematical developments can be limited to a reasonable extent.

### 3 Variational formulation

The main objective of this work is to find some simple sufficient conditions under which Problem 1 is well posed (i.e., it has a unique solution which continuously depends on the data) and its solution can be approximated by a Galerkin finite element method. In the above statement we mean by “simple conditions” those entailing the aimed results through Lax-Milgram and first Strang lemmas. The first step towards our goal is to obtain from Problem 1 a variational formulation.

First of all, let us state and prove the following lemma

**Lemma 1.** Any solution of Problem 1 is fully determined by its \( E \) component through

\[
B = -\frac{1}{j\omega} J_m - \frac{1}{j\omega} \text{curl} E, \quad (6)
\]

\[
H = M E - \frac{c}{j\omega} Q J_m - \frac{c}{j\omega} Q \text{curl} E, \quad (7)
\]

and

\[
D = \frac{1}{c} P E - \frac{1}{j\omega} L J_m - \frac{1}{j\omega} L \text{curl} E. \quad (8)
\]

**Proof.** Suppose \((E, B, H, D)\) satisfies Problem 1. Then, as \(\omega > 0\), from (5) we deduce (6), which substituted for \(B\) in (1) gives (7) and (8). \(\square\)

Now we exploit Lemma 1 to express the remaining (i.e., not used in deducing Lemma 1 itself) equations of Problem 1 by \(E\) only. In order to do so, first we take the \(L^2\) scalar product of (5)1 with any vector field \(v \in U\):

\[
(\text{curl} H - j\omega D, v)_{0,\Omega} = (J_e, v)_{0,\Omega} \quad \forall v \in U. \quad (9)
\]

Then, owing to H1-H2, we can exploit the Green formula (72) (see Appendix A)

\[
(c_{\text{curl}} u, v)_{0,\Omega} = (u, \text{curl} v)_{0,\Omega} - (u \times n, n \times v \times n)_{0,\Gamma} \quad \forall u, v \in U \quad (10)
\]

to get

\[
(H, \text{curl} v)_{0,\Omega} - j\omega (D, v)_{0,\Omega} - (H \times n, n \times v \times n)_{0,\Gamma} = (J_e, v)_{0,\Omega} \quad \forall v \in U. \quad (11)
\]

By substituting (7), (8) and (5)3 in (11), while taking into account H3-H4, we obtain:

\[
(M E, \text{curl} v)_{0,\Omega} - \frac{c}{j\omega} (Q \text{curl} E, \text{curl} v)_{0,\Omega} - \frac{j\omega}{c} (P E, v)_{0,\Omega}
\]

\[
- (Y (n \times E \times n), n \times v \times n)_{0,\Gamma} + (L \text{curl} E, v)_{0,\Omega}
\]

\[
= (J_e, v)_{0,\Omega} + (f_R, n \times v \times n)_{0,\Gamma} + \frac{c}{j\omega} (Q J_m, \text{curl} v)_{0,\Omega} - (L J_m, v)_{0,\Omega} \quad \forall v \in U. \quad (12)
\]
Thus, by defining the sesquilinear form
\[
a(u, v) = c(Q \text{curl} u, \text{curl} v)_{0, \Omega} - \frac{\omega^2}{c} (P u, v)_{0, \Omega} - j\omega (M u, \text{curl} v)_{0, \Omega} - j\omega (L \text{curl} u, v)_{0, \Omega} + j\omega (Y (n \times u \times n), n \times v \times n)_{0, \Gamma}
\]
and the antilinear form
\[
l(v) = -j\omega (J_e, v)_{0, \Omega} - c(Q J_m, \text{curl} v)_{0, \Omega} + j\omega (L J_m, v)_{0, \Omega} - j\omega (f_R, n \times v \times n)_{0, \Gamma}
\]
the variational formulation we were looking for is

**Problem 2.** Under the hypotheses H1-H4, given \( \omega > 0 \), \( J_e \in (L^2(\Omega))^3 \), \( J_m \in (L^2(\Omega))^3 \) and \( f_R \in L^2_2(\Gamma) \), find \( E \in U \) such that
\[
a(E, v) = l(v) \quad \forall v \in U
\]
where our hypotheses are sufficient to give a meaning to all terms.

As a matter of fact, in deducing Problem 2 from Problem 1, we have shown that the \( E \) component of any solution of Problem 1 satisfies Problem 2.

Now, we prove that any solution of Problem 2 is the \( E \) component of a solution of Problem 1, the other components of which are given by (6)-(8). Suppose \( E \) satisfies Problem 2. Owing to Lemma 1, \( (E, B, H, D) \) cannot satisfy Problem 1 unless \( B, H \) and \( D \) are obtained by (6), (7) and (8). Hence, taking into account that owing to H3 all the coefficients of (6), (7) and (8) are bounded, we have \( (E, B, H, D) \in U \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \), satisfying (1) and (5)2.

By taking \( v \in H_0(\text{curl}, \Omega) = \{ v \in H(\text{curl}, \Omega) \mid v \times n = 0 \text{ on } \Gamma \} \subset U \) in (15) (i.e. (12)) and back-substituting (7) and (8) we get
\[
(H, \text{curl} v)_{0, \Omega} = j \omega (D, v)_{0, \Omega} + (J_e, v)_{0, \Omega} \quad \forall v \in H_0(\text{curl}, \Omega).
\]

Let us denote by \( D(\Omega) \) the space of infinitely differentiable functions with compact support in \( \Omega \) and values in \( \mathbb{C} \) [27] (p. 457), [28] (p. 28), by \( D'(\Omega) \) its dual space of distributions on \( \Omega \) [27] (p. 463), [28] (p. 48), and by \( (D'(\Omega))^3 \langle u, v \rangle_{D(\Omega)^3} \) the antiduality pairing between \( (D'(\Omega))^3 \) and \( (D(\Omega))^3 \). Since \( (D(\Omega))^3 \subset H_0(\text{curl}, \Omega) \), by taking \( v \in (D(\Omega))^3 \) and exploiting the definition of the derivative of a distribution [27] (p. 467), [28] (p. 49), we obtain
\[
(D'(\Omega))^3 \langle \text{curl} H, v \rangle_{(D(\Omega))^3} = (j \omega D + J_e, v)_{0, \Omega} \quad \forall v \in (D(\Omega))^3.
\]

Hence, (5)1 holds in the sense of distributions on \( \Omega \). As the right-hand side makes sense for \( v \in (L^2(\Omega))^3 \) and \( (D(\Omega))^3 \) is dense in \( (L^2(\Omega))^3 \), the left-hand side uniquely extends by continuity to a functional on \( (L^2(\Omega))^3 \), which we identify with an element of \( (L^2(\Omega))^3 \) by the Riesz representation theorem [27] (p. 302). Thus, \( \text{curl} H \in (L^2(\Omega))^3 \) and the antiduality pairing reduces to the scalar product in that space. Hence, (5)1 holds as an equality in \( (L^2(\Omega))^3 \) and \( H \in H(\text{curl}, \Omega) \).

Starting again from (15), this time with \( v \in (H^1(\Omega))^3 \), and working out the same back-substitutions as before, we obtain
\[
(H, \text{curl} v)_{0, \Omega} = j \omega (D, v)_{0, \Omega} - (Y (n \times E \times n), n \times v \times n)_{0, \Gamma} = (J_e, v)_{0, \Omega} + (f_R, n \times v \times n)_{0, \Gamma} \quad \forall v \in (H^1(\Omega))^3.
\]

As \( H \in H(\text{curl}, \Omega) \) with \( \Omega \) satisfying H1-H2, we can apply the Green formula (69) (see Appendix A) to obtain
\[
\text{curl} H, v)_{0, \Omega} = (H, n \times v \times n)_{\frac{1}{2}, \Omega} - j \omega (D, v)_{0, \Omega} - (Y (n \times E \times n), n \times v \times n)_{0, \Gamma} = (J_e, v)_{0, \Omega} + (f_R, n \times v \times n)_{0, \Gamma} \quad \forall v \in (H^1(\Omega))^3.
\]
which, owing to the fact that \((5)_1\) is satisfied in \((L^2(\Omega))^3\), reduces to
\[
\langle H \times n, v \rangle_{\frac{1}{2}, \Gamma} = \langle Y (n \times E \times n), n \times v \times n \rangle_{0, \Gamma} + \langle f_R, n \times v \times n \rangle_{0, \Gamma} \quad \forall v \in (H^1(\Omega))^3.
\]  
(20)

Since the right-hand side of (20) depends on \(v\) through \(n \times v \times n\) only, the left-hand side also does and, owing to (63) (see Appendix A), actually is a functional on \(V_\pi\) and, therefore, it can be written as \(v_B^* \langle H \times n, n \times v \times n \rangle_{V_\pi}\). Then, by exploiting again (63), it follows that
\[
\langle v_B^* (H \times n, n \times v \times n) \rangle_{V_\pi} = \langle Y (n \times E \times n), q \rangle_{0, \Gamma} + \langle f_R, q \rangle_{0, \Gamma} \quad \forall q \in V_\pi.
\]  
(21)

Owing to H4, the right-hand side is a \(L^2(\Gamma)\) scalar product. Furthermore, it makes sense for \(q \in L^2(\Gamma)\). Hence, by the density of \(V_\pi\) in \(L^2(\Gamma)\) [29] (p. 850), the left-hand side uniquely extends by continuity to a functional on \(L^2(\Gamma)\), which we identify with an element of \(L^2(\Gamma)\).

Then, \(H \times n \in L^2(\Gamma)\) and \((5)_3\) follows. Therefore, \(H \in U\) and \((E, B, H, D)\) satisfies Problem 1.

Hence, we have proved the following theorem.

**Theorem 1.** If \((E, B, H, D)\) satisfies Problem 1, then \(E\) satisfies Problem 2. Conversely, if \(E\) satisfies Problem 2 and \(B, H\) and \(D\) are obtained by (6)-(8), then \((E, B, H, D)\) satisfies Problem 1.

Suppose now that Problem 2 is well-posed. Then, existence and uniqueness for Problem 1 immediately follows from Theorem 1.

Continuous dependence on data for Problem 2 reads
\[
\|E\|_{U, \Omega} \leq C_1 \|J_c\|_{0, \Omega} + C_2 \|J_m\|_{0, \Omega} + C_3 \|f_R\|_{0, \Gamma}
\]  
(22)

and entails that also \(\|E\|_{0, \Omega}, \|\text{curl} E\|_{0, \Omega}\) and \(\|E \times n\|_{0, \Gamma}\) are controlled by the same right-hand side. Hence, taking into account that owing to H3 all the coefficients of (6), (7) and (8) are bounded, similar estimates for \(\|B\|_{0, \Omega}, \|D\|_{0, \Omega}\) and \(\|H\|_{0, \Omega}\) follow from (6), (7), (8) and the triangle inequality. Finally, taking into account H4 and that \(\|n \times E \times n\|_{0, \Gamma} = \|E \times n\|_{0, \Gamma}\), (5) and (5)_3 give the estimates for \(\|\text{curl} H\|_{0, \Omega}\) and \(\|H \times n\|_{0, \Gamma}\) and, thus, an estimate for \(\|H\|_{U, \Omega}\) holds, too. Hence the solution of Problem 1 continuously depends on data.

We have then proved that well-posedness of Problem 2 implies well-posedness of Problem 1. The converse is trivial because existence and uniqueness is again immediately obtained from Theorem 1, while continuous dependence on data for Problem 1 include (22).

Hence the following theorem holds true

**Theorem 2.** Problem 1 is well-posed if and only if Problem 2 is well-posed.

4 Further hypotheses for well posedness

Following the same approach we used in [6], we will look for conditions that allow to apply the Lax-Milgram theorem [30], [27] (pp. 376-377), by which the well posedness of Problem 2 follows under the continuity and the coercivity of the sesquilinear form (13) and the continuity of the antilinear form (14). From this point of view, in some way, this section is a simple analysis of the conditions which guarantee that the sesquilinear form (13) is coercive, being the two continuity conditions almost trivial.

In fact, the following theorem can be easily proved.
Theorem 3. Whenever \( \omega > 0 \) and H1-H4 are satisfied, the sesquilinear form \( a(u,v) \) is continuous on \( U \times U \) and the antilinear form \( l(v) \) is continuous on \( U \) if \( J_e \in (L^2(\Omega))^3 \), \( J_m \in (L^2(\Omega))^3 \) and \( f_R \in L^2(\Gamma) \).

**Proof.** The hypotheses H1 and H2 are necessary to give a precise meaning to the space \( U \) [20]. Owing to H3, \( P \) is bounded. Hence, \( P u \in (L^2(\Omega))^3 \) and \( \exists C_1 > 0 \) such that \( \| P u \|_{0,\Omega} \leq C_1 \| u \|_{0,\Omega} \). By using again H3 and H4, we get in the same way \( \| Q \text{curl} u \|_{0,\Omega} \leq C_2 \| u \|_{0,\Omega} \), \( \| J_m \|_{0,\Omega} \leq C_2 \| J_m \|_{0,\Omega} \), \( \| Y (n \times u \times n) \|_{0,\Gamma} \leq C_3 \| n \times u \times n \|_{0,\Gamma} \), \( \| M u \|_{0,\Omega} \leq C_4 \| u \|_{0,\Omega} \), \( \| L \text{curl} u \|_{0,\Omega} \leq C_5 \| u \|_{0,\Omega} \) and \( \| L J_m \|_{0,\Omega} \leq C_5 \| J_m \|_{0,\Omega} \), where all the constants are positive. Both continuities are now a consequence of the Cauchy-Schwartz inequality. \( \square \)

Before introducing some additional hypotheses which can guarantee the coercivity of (13), let us point out that in the case of anisotropic materials (i.e., \( L \)), Theorem 3. of [6] will be useful to show the link between our hypotheses and those in [6].

Now, let us introduce a more compact 6-by-6 matrix notation [21] that will be widely exploited in the following. As a matter of fact (13) can be recast in the form

\[
a(u,v) = \int_{\Omega} \left\{ (v^*, \text{curl} v^*) \begin{pmatrix} -\omega^2 P & -j \omega L \\ -j \omega M & cQ \end{pmatrix} \begin{pmatrix} u \\ \text{curl} u \end{pmatrix} + j \omega (Y n \times u \times n, n \times v \times n)_{0,\Gamma} \right\}
\]

Then, we can define

\[
A = \begin{pmatrix} -\omega^2 P & -j \omega L \\ -j \omega M & cQ \end{pmatrix}
\]

and rewrite it as \( A = B - jC \), where both

\[
B = \frac{A + A^*}{2} = \begin{pmatrix} -\omega^2 P & \omega L - M^* \\ \omega M - L^* & cQ/2 \end{pmatrix}
\]

and

\[
C = \frac{A^* - A}{2j} = \begin{pmatrix} -\omega^2 P & \omega M^* + L \\ \omega M + L^* & cQ/2 \end{pmatrix}
\]

are hermitian matrices.

In order to impose that all media in \( \Omega \) are passive, we assume H1-H3 and the sufficient condition [21], [31], [32], [33]

**H5.** \( p^* c p \leq 0 \), \( \forall p \in C^6 \), \( \forall x \in \Omega_i \), \( \forall i \in I \)

which for anisotropic materials reduces to the corresponding hypothesis (i.e., H7) of [6], which reads

\[
v^* \frac{\varepsilon^* - \varepsilon}{2j} v \geq 0 \text{ and } v^* \frac{\mu^* - \mu}{2j} v \geq 0, \forall v \in C^3 \text{ and } \forall x \in \Omega_i, i \in I.
\]

This can be easily seen from (28), (23), (24) and Theorem 1 of [6]. Notice that in the regions where the equality holds true the materials are lossless.

Again, as in [6], we decided to consider just the following simpler assumption of a “everywhere lossy boundary”
H6. \( \exists \alpha \in \mathbb{R}, \alpha > 0, \) such that \( \text{Re}(Y(x)) \geq \alpha \) almost everywhere on \( \Gamma, \)

since it is sufficient to cover both the lowest order absorbing boundary condition and boundary conditions at imperfectly conducting surfaces.

Let us define for convenience the six component column vector

\[
w = w(u) = \begin{pmatrix} u \\ \text{curl} u \end{pmatrix},
\]

Looking at the expression for \( |a(u, u)| \) (see (84)), it is immediately apparent that H6 and

\[
\exists K > 0 \text{ such that } \int_{\Omega} w^* C w \leq -K (\|u\|_{0, \Omega}^2 + \|\text{curl} u\|_{0, \Omega}^2) \quad \forall u \in U,
\]

are sufficient conditions for the coercivity of (13).

However, this is not a very interesting result. In fact, it covers only those particular situations in which just \( \text{Im}(a(u, u)) \) alone is sufficient to control the \( U \) norm. In the general case, instead, both the real and the imaginary parts of the sesquilinear form cooperate to achieve this control, which, on the other hand, needs not be obtained in the same way in each of the subregions of \( \Omega \) that are filled with different materials. Hence, we will consider combinations of more selective conditions similar to (31), but holding for some subregion \( \Omega_s \) of \( \Omega \) and involving either \( B \) or \( C \) or submatrices of them and either \( \|u\|_{0, \Omega_s} \) or \( \|\text{curl} u\|_{0, \Omega_s} \) or both.

In Section 6 some examples taken from the open literature will show that these considerations allow a significant extension of the theory with respect to the one based just on (31).

Due to H5, the role of matrices \( B \) and \( C \) is not the same. Moreover, the importance of the losses for the conclusions we are interested in is well known. For these reasons we introduce the indicated set of more selective conditions starting from those involving the matrix \( C \) alone.

The first condition we consider, which is apparently weaker than (31), is

\[
\int_{\Omega_{el}} w^* C w \leq -K_{el} \|u\|_{0, \Omega_{el}}^2 \quad \forall u \in H(\text{curl}, \Omega_{el})
\]

for some \( K_{el} > 0 \) and some \( \Omega_{el} \subset \Omega \).

A condition like (32), however, can be regarded as a property of the materials filling \( \Omega_{el} \) rather than a property of a particular configuration of them, if and only if it holds true \( \forall \Omega \subseteq \Omega_{el} \). Hence, it is convenient to strengthen this condition in this way, in order to get our main result in terms of material properties. This makes more easily understandable the material configurations we are dealing with. Doing so, (32) is substituted by the following condition implying it:

\[
\int_{\Omega} (w^* C w + K_{el} |u|^2) \leq 0 \quad \forall u \in H(\text{curl}, \Omega) \quad \forall \Omega \subseteq \Omega_{el}
\]

for some \( K_{el} > 0 \) and some \( \Omega_{el} \subset \Omega \).

Since we have imposed that all media are passive through H5, a sufficient condition expressed by an algebraic property of \( C \) that holds almost everywhere in \( \Omega \), in the same spirit we look for a similar condition implying (33). Therefore, after posing

\[
p = \begin{pmatrix} q \\ r \end{pmatrix},
\]

where \( q, r \in \mathbb{C}^3 \) and, thus, \( p \in \mathbb{C}^6 \), we define \( \Omega_{el} \) as the union of the subdomains \( \Omega_i \) of \( \Omega \) such that \( \exists K_{el} > 0 \) such that

\[
p^* Cp \leq -K_{el} |q|^2 \quad \forall p \in \mathbb{C}^6, \quad \forall x \in \Omega_i.
\]

Since (35) implies (33), it ensures the control of the \( L^2 \) norm in \( \Omega_{el} \).

In the case of anisotropic materials (35) reduces to

\[
\begin{cases}
\frac{\omega^2}{c^2} \mathbf{v} \cdot \mathbf{P}^* \mathbf{E} \mathbf{v} \geq K_{el} |\mathbf{v}|^2 \\
\mathbf{v} \cdot \mathbf{Q}^* \mathbf{Q} \mathbf{v} \leq 0
\end{cases}
\]

where \( \mathbf{P}, \mathbf{Q} \in \mathbb{C}^{3 \times 3} \) and \( \omega, c \in \mathbb{R} \).
for all $v \in \mathbb{C}^3$ and for all $x \in \Omega_i$ (see Appendix B).

By exploiting (23), (24) and Theorem 1 of [6], it is easily shown that, under the assumption $H7$ of [6] (see (29)), (36) is redundant and $\Omega_{el}$ coincides with $D_{el}$ of [6].

In a similar way, we define $\Omega_{ml} \subseteq \Omega$ as the union of the subdomains $\Omega_i$ of $\Omega$ such that $\exists K_{ml} > 0$ such that

$$p^*Cp \leq -K_{ml}|r|^2 \quad \forall p \in \mathbb{C}^6, \forall x \in \Omega_i,$$

which implies that

$$\int_{\Omega} w^*Cw \leq -K_{ml}||u||^2_{0,\Omega}$$

for all $u \in H(curl,\tilde{\Omega})$ and $\forall \Omega \subseteq \Omega_{ml}$, so ensuring the control of the curl seminorm in $\Omega_{ml}$.

In the case of anisotropic materials (37) reduces to

$$\left\{ \begin{array}{l}
\epsilon_v^*\frac{Q_{\omega}^*}{2}\v \leq K_{ml}|v|^2 \\
\epsilon_v^*\frac{P_{\omega}^*}{2}\v \geq 0
\end{array} \right.$$  (39)

for all $\v \in \mathbb{C}^3$ and for all $x \in \Omega_i$ (see Appendix B).

Arguing as before we conclude that $\Omega_{el}$ reduces to $D_{el}$ of [6].

Notice that both definitions are consistent with the assumption $H5$ and make it stronger in both $\Omega_{el}$ and $\Omega_{ml}$.

As we have already seen, in the particular case of anisotropic materials, $\Omega_{el}$ and $\Omega_{ml}$ respectively reduce to $D_{el}$ and $D_{ml}$ of [6], which are the region filled with materials having electric losses and the region filled with materials having magnetic losses, respectively. While also for bianisotropic materials magnetic losses may vanish everywhere in $\Omega_{el}$ and electric losses may vanish everywhere in $\Omega_{ml}$, in the most general case of them (i. e., when also $C$ has nonzero off diagonal submatrices) a certain amount of magnetic losses is required in $\Omega_{el}$ and, likewise, some electric losses are required in $\Omega_{ml}$. This is explained again in Appendix B.

Now we define a “double lossy” subregion $\Omega_{dl} \subseteq \Omega$, as the union of the subdomains $\Omega_i$ of $\Omega$ for which $\exists K_{dl} > 0$ such that

$$p^*Cp \leq -K_{dl}|p|^2 \quad \forall p \in \mathbb{C}^6, \forall x \in \Omega_i.$$  (40)

From (40) it follows that the control of the curl norm is readily achieved in $\Omega_{dl}$.

Notice that, as shown in Appendix B, we have $\Omega_{dl} = \Omega_{el} \cap \Omega_{ml}$.

Then, we define $\Omega_{ll} = \Omega \setminus (\Omega_{el} \cup \Omega_{ml} \cup \Omega_{dl})$.

Even though some losses may be present in this subregion (i. e., $p^*Cp$ is not zero), they do not satisfy any condition useful for the aimed coercivity proof (i. e., none of relevant norms or seminorms is controlled). Hence, from this viewpoint, this subregion can be regarded as essentially “lossless”.

The set of more selective conditions involving the matrix $C$ alone is complete and we switch our attention to the definitions of conditions involving the matrix $B$ alone. $\Omega_{en} \subseteq \Omega$ is the union of the subdomains $\Omega_i \subseteq \Omega$ for which $\exists K_{en} > 0$ such that

$$\frac{\omega^2}{c}q^*P^* + \frac{P}{2}q \leq -K_{en}|q|^2 \quad \forall q \in \mathbb{C}^3, \forall x \in \Omega_i.$$  (41)

By exploiting again (23), (24) and Theorem 1 of [6], it can be seen that (41) is just the condition involved in the definition of $D_{en}$ of [6], but we have precisely $D_{en} = \Omega_{en} \setminus \Omega_{el}$. Similar considerations hold true for $\Omega_{ep}, \Omega_{mp}$ and $\Omega_{mn}$ of which we will give only the definitions.

$\Omega_{ep} \subseteq \Omega$ is the union of the subdomains $\Omega_i \subseteq \Omega$ for which $\exists K_{ep} > 0$ such that

$$\frac{\omega^2}{c}q^*P^* + \frac{P}{2}q \geq K_{ep}|q|^2 \quad \forall q \in \mathbb{C}^3, \forall x \in \Omega_i.$$  (42)

$\Omega_{mp} \subseteq \Omega$ is the union of the subdomains $\Omega_i \subseteq \Omega$ for which $\exists K_{mp} > 0$ such that

$$c\epsilon_r^*\frac{Q^*}{2}r \geq K_{mp}|r|^2 \quad \forall r \in \mathbb{C}^3, \forall x \in \Omega_i.$$  (43)
\[\Omega_{nn} \subseteq \Omega \text{ is the union of the subdomains } \Omega_i \subseteq \Omega \text{ for which } \exists K_{nn} > 0 \text{ such that} \]
\[c r^* Q^* + \frac{Q}{2} \leq -K_{nn} |r|^2 \quad \forall r \in \mathbb{C}^3, \forall x \in \Omega_i. \tag{44}\]

Finally, we define a \(B\)-positive subregion \(\Omega_{B_p}\) and a \(B\)-negative subregion \(\Omega_{B_n}\), as follows:
\[\Omega_{B_p} \subseteq \Omega \text{ is the union of the subdomains } \Omega_i \subseteq \Omega \text{ for which } \exists K_{B_p} > 0 \text{ such that} \]
\[p^* B p \geq K_{B_p} |p|^2 \quad \forall p \in \mathbb{C}^6, \forall x \in \Omega_i, \tag{45}\]
\[\Omega_{B_n} \subseteq \Omega \text{ is the union of the subdomains } \Omega_i \subseteq \Omega \text{ for which } \exists K_{B_n} > 0 \text{ such that} \]
\[p^* B p \leq -K_{B_n} |p|^2 \quad \forall p \in \mathbb{C}^6, \forall x \in \Omega_i. \tag{46}\]

It is apparent that either (45) or (46) entails the control of the curl norm and should hold in \(\Omega_{ll}\), where \(C\) gives no control at all, if the aimed coercivity is to be achieved. It is less obvious, yet true, that (41)-(44) are just the kind of conditions we need to supplement the partial control given by \(C\) in \(\Omega_{el} \setminus \Omega_{el}\) and \(\Omega_{el} \setminus \Omega_{ll}\).

Since in this case, unlike the similar situation involving \(C\), the conditions that hold true in \(\Omega_{en} \cap \Omega_{en}\) and in \(\Omega_{ep} \cap \Omega_{en}\) are weaker than required to control the curl norm, we have \(\Omega_{B_p} \subseteq \Omega_{mp} \cap \Omega_{en}\) and \(\Omega_{B_n} \subseteq \Omega_{en} \cap \Omega_{mn}\).

As the reader should have understood at this point, the above definitions are not arbitrarily chosen. They are just the ones that allow to work out the same kind of proof of Theorem 3 of [6] in the more general context of bianisotropic metamaterials.

Now, we are in a position to state the following theorem, which is proved in Appendix C.

**Theorem 4.** Whenever \(\omega > 0\), \(H1-H6\) are satisfied, the sesquilinear form \(a(u,v)\) is coercive on \(U\) if condition

\[
H7. \quad \left[(\Omega_{el} \setminus \Omega_{ll} \subseteq \Omega_{mp}) \text{ and } (\Omega_{el} \setminus \Omega_{ll} \subseteq \Omega_{en}) \text{ and } (\Omega_{ll} \subseteq \Omega_{B_p})\right] \text{ or } \left[(\Omega_{el} \setminus \Omega_{ll} \subseteq \Omega_{mn}) \text{ and } (\Omega_{el} \setminus \Omega_{ll} \subseteq \Omega_{en}) \text{ and } (\Omega_{ll} \subseteq \Omega_{B_n})\right]
\]

holds true.

Then, by using the Lax-Milgram Theorem and Theorems 3 and 4, we conclude that:

**Theorem 5.** Whenever \(H5-H7\) are satisfied, Problem 2 is well posed.

Then, the following theorem is an easy consequence of Theorem 5, Theorem 2 and Theorem 1.

**Theorem 6.** Whenever \(H5-H7\) are satisfied, Problem 1 is well posed and its solution can be obtained from the solution of Problem 2 as in Theorem 2.

**Remark 1.** Since \(\Omega_{ll} = \Omega_{ml} \cap \Omega_{el}\) (see Appendix B), \(H7\) is equivalent to
\[
\left[(\Omega_{el} \setminus \Omega_{ml} \subseteq \Omega_{mp}) \text{ and } (\Omega_{ml} \setminus \Omega_{el} \subseteq \Omega_{en}) \text{ and } (\Omega_{ll} \subseteq \Omega_{B_p})\right] \text{ or } \left[(\Omega_{el} \setminus \Omega_{ml} \subseteq \Omega_{mn}) \text{ and } (\Omega_{ml} \setminus \Omega_{el} \subseteq \Omega_{en}) \text{ and } (\Omega_{ll} \subseteq \Omega_{B_n})\right]
\]

If all materials involved are just anisotropic, rather than bianisotropic, we have also \(\Omega_{B_p} = \Omega_{en} \cap \Omega_{mp}\) and \(\Omega_{B_n} = \Omega_{en} \cap \Omega_{ep}\). Hence, the first term of the disjunction in \(H7\) becomes \(\left[(\Omega_{el} \setminus \Omega_{ml} \subseteq \Omega_{mp}) \text{ and } (\Omega_{ml} \setminus \Omega_{el} \subseteq \Omega_{en}) \text{ and } (\Omega_{ll} \subseteq \Omega_{en} \cap \Omega_{mp})\right]\), which is equivalent to \(\Omega \setminus \Omega_{el} \subseteq \Omega_{mp}\) and \(\Omega \setminus \Omega_{el} \subseteq \Omega_{en}\).

Dealing in the same way with the second term, \(H7\) reduces to
\[
\left[(\Omega \setminus \Omega_{ml} \subseteq \Omega_{mp}) \text{ and } (\Omega \setminus \Omega_{el} \subseteq \Omega_{en})\right] \text{ or } \left[(\Omega \setminus \Omega_{ml} \subseteq \Omega_{mn}) \text{ and } (\Omega \setminus \Omega_{el} \subseteq \Omega_{en})\right],
\]
which is recognized as equivalent to H9 of [6], which can be written also as

\[ \{ \Omega \setminus D_{ml} = D_{mp} \} \text{ and } \{ \Omega \setminus D_{el} = D_{cn} \} \text{ or } \{ (\Omega \setminus D_{ml} = D_{mn}) \text{ and } (\Omega \setminus D_{el} = D_{ep}) \}, \]

if we remember that, owing to slightly different definitions, \( D_{xy} \) of [6] is related to the corresponding subregion \( \Omega_{xy} \) of the present paper by \( D_{ml} = \Omega_{ml}, D_{el} = \Omega_{el}, D_{mp} = \Omega_{mp}\setminus \Omega_{ml}, D_{mn} = \Omega_{mn}\setminus \Omega_{ml}, D_{cn} = \Omega_{cn}\setminus \Omega_{el} \text{ and } D_{ep} = \Omega_{ep}\setminus \Omega_{el}. \)

**Remark 2.** As the reader could see looking at Appendix C, the proof of Theorem 4 would work even if \( \Omega_{el} \) were defined by (32), rather than by (35), and all the other subregions of \( \Omega \) involved in H7 were defined accordingly. We mention just for the sake of completeness that this more general version of Theorem 4 is feasible, but we have no practical example fitting it. As a matter of fact, in general, is more convenient to work with conditions like (35), which can be thought as material properties, rather than with conditions like (32), which express properties of the specific configuration under consideration and are more difficult to check.

### 5 Galerkin and finite element approximability

Proving the convergence of Galerkin and finite element approximations of Problem 2 is now standard matter and can be worked out precisely as in sections 5 and 6 of [6] by exploiting the first Strang lemma.

Let us spend, instead, some words about the approximability of Problem 1.

Suppose that, as in [6], from an approximation of Problem 2 we have obtained \( E_h \rightarrow E \) in \( U \) as \( h \rightarrow 0 \) (i.e., \( \lim_{h \rightarrow 0} \| E - E_h \|_{U; \Omega} = 0 \)), under the assumptions (among others) that the approximate data \( J_{eh}, J_{mh} \) and \( f_{Rh} \) tend to the exact ones \( J, J_m \) and \( f_R \) in \( (L^2(\Omega))^3 \) or \( (L^2(\Omega))^3 \) as \( h \rightarrow 0 \). It should be noticed that in [6] we assumed to use Neelede’s edge elements, which are curl-conforming and, thus, most suitable to approximate fields belonging to \( H(\text{curl}, \Omega) \). However, this is not mandatory to achieve convergence because any reasonable finite element does satisfy the approximation property needed to apply the first Strang lemma (i.e., H14 of [6]).

If we define \( B_h, H_h \) and \( D_h \) as functions of \( E_h \) and \( J_{mh} \) through formulas that exactly parallel (6), (7) and (8), then we get \( B_h \rightarrow B, H_h \rightarrow H \) and \( D_h \rightarrow D \) in \( (L^2(\Omega))^3 \) as \( h \rightarrow 0 \). Notice that, even if \( H \in U \), in general, \( H_h \) does not converge to \( H \) in \( U \) because, in general, \( H_h \notin H(\text{curl}, \Omega) \).

Hence, the convergence of \( (E_h, B_h, H_h, D_h) \) to the solution \( (E, B, H, D) \) of Problem 1 takes place in \( U \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \) even if \( (E, B, H, D) \in U \times (L^2(\Omega))^3 \times U \times (L^2(\Omega))^3 \) and Problem 1 is well posed in this space.

If, moreover, \( J_{eh} \rightarrow J_e \) and \( J_{mh} \rightarrow J_m \) in \( H(\text{div}, \Omega) \) as \( h \rightarrow 0 \) and curl-conforming elements are used to approximate \( E_h \), then \( B_h \rightarrow B \) in \( H(\text{div}, \Omega) \), but a similar result does not hold, in general, for \( D_h \), since it does not necessarily belong to \( H(\text{div}, \Omega) \).

Hence, \( (E_h, B_h, H_h, D_h) \rightarrow (E, B, H, D) \) in \( U \times H(\text{div}, \Omega) \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \) as \( h \rightarrow 0 \) even if, in this case, \( (E, B, H, D) \in U \times H(\text{div}, \Omega) \times U \times H(\text{div}, \Omega) \) and Problem 1 is well posed in this space. Notice that, if the finite elements used to approximate \( E_h \) are not curl-conforming, then the strong convergence of \( J_{eh} \) and \( J_{mh} \) is not exploited at all and \( (E_h, B_h, H_h, D_h) \rightarrow (E, B, H, D) \) in \( U \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \times (L^2(\Omega))^3 \), as before.

### 6 Practical implications

In this section we would like to show that the developed theory can be used to prove, for the first time to the best of authors’ knowledge, the well posedness and the finite element approximability of electromagnetic models involving bianisotropic media of practical interest. It will be pointed out that the same results could not be obtained with the simpler version of the theory based just on (31).

The examples considered could also be useful to show how such a theory can be exploited. In particular, how one can check the validity of the crucial condition H7 in cases of practical interest.
However, before considering such examples, it can be useful to point out that some models of bianisotropic media of practical interest do not allow the application of our theory. For example consider the bianisotropic medium studied in [34]. The constitutive relations are

\[
\begin{align*}
D &= \varepsilon_r \varepsilon_0 I E - j \xi_c I B \\
H &= -j \xi_c I E + \frac{1}{\mu_0 \mu_r} I B
\end{align*}
\]

where \(\varepsilon_r, \mu_r, \text{ and } \xi_c\) are strictly positive real quantities. Thus, from (1) we can identify

\[
P = \sqrt{\frac{\varepsilon_0}{\mu_0}} \varepsilon_r I
\]

\[
L = M = -j \xi_c I
\]

\[
Q = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{1}{\mu_r} I
\]

so that \(P^* - P = 0, Q^* - Q = 0, L^* + M^* = 0\) and \(L^* + M = 0\). Thus, in particular, \(C = 0\) and the region occupied by this medium is contained in \(\Omega_{ll}\). Moreover, \(P^* + P, Q^* + Q = 0\) and \(P\) and \(Q\) are obviously positive definite. Then, according to our definitions, this medium is contained in \(\Omega_{ep}\) and \(\Omega_{mp}\) and, consequently, it cannot be contained neither in \(\Omega_{Bp}\) nor in \(\Omega_{Bn}\). Thus, the inclusions \(\Omega_{ll} \subseteq \Omega_{Bp}\) and \(\Omega_{ll} \subseteq \Omega_{Bn}\) are both violated and condition \(H7\) cannot hold true, independently of the value of \(\xi_c \in \mathbb{R}\) and independently of any other material involved in the model of interest.

We conclude that our theory cannot be applied but note that so far, to the best of authors’ knowledge, there is no way to guarantee the well posedness and the finite element approximability for any time-harmonic model involving such a medium.

On the contrary, for the medium studied by Maruyama and Koshiba [35] we have

\[
\begin{align*}
D &= \varepsilon E - j \xi B \\
H &= -j \eta E + \mu^{-1} B
\end{align*}
\]

where

\[
\varepsilon = \varepsilon_0 1.1 (1 - j 0.01) I
\]

\[
\mu = \mu_0 \begin{bmatrix} 1 & j \mu_a & 0 \\ -j \mu_a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mu_a \in \mathbb{R},
\]

\[
\xi = \eta = 0.001 I.
\]

Therefore, from (1) we can identify

\[
P = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{1}{\mu_0} 1.1 (1 - j 0.01) I
\]

\[
L = M = -j 0.001 I
\]

and, when \(|\mu_a| \neq 1\) (otherwise the matrix \(\mu\) is singular)

\[
Q = \sqrt{\frac{\varepsilon_0}{\mu_0}} \frac{1}{1 - \mu_a^2} \begin{bmatrix} 1 & -j \mu_a & 0 \\ j \mu_a & 1 & 0 \\ 0 & 0 & 1 - \mu_a^2 \end{bmatrix}.
\]

Thus, \(L + M^* = 0, L^* + M = 0\) and \(Q^* - Q = 0\) whereas \(P^* - P = \sqrt{\frac{\varepsilon_0}{\mu_0}} j 0.022 I\) and we deduce that the \(6 \times 6\) matrix \(C\) is

\[
C = -\omega^2 \varepsilon_0 0.011 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.
\]
Then, according to our definitions, the region $\Omega_{MK}$ occupied by this medium is contained in $\Omega_{el}$ and then it cannot be a part of $\Omega_{ll}$. Moreover, it is not a part of $\Omega_{ml}$ and then it is not contained in $\Omega_{dp}$. One can note that when such a medium is involved in the model of interest the results of well posedness and finite element approximability cannot be obtained with the simplest version of the theory based just on inequality (31), since necessarily it does not hold true. Fortunately, as we are going to show, infinite many models involving such a medium can be managed by using the more sophisticated theory developed in this paper. In order to obtain such a result, let us consider the definitions involving the matrix $B$ or its submatrices. We have

$$
-\frac{\omega^2}{c} \frac{P + P^*}{2} = -\omega^2 \varepsilon_0 1.1 I \tag{59}
$$

$$
\frac{\omega}{2j} \frac{L - M^*}{2j} = \omega \frac{M - L^*}{2j} = -\omega 0.001 I \tag{60}
$$

$$
\frac{c}{2} \frac{Q + Q^*}{2} = \frac{1}{\mu_0(1 - \mu_a^2)} \begin{bmatrix}
1 & -j\mu_a & 0 \\
-j\mu_a & 1 & 0 \\
0 & 0 & 1 - \mu_a^2
\end{bmatrix} \tag{61}
$$

Notice that, if $|\mu_a| < 1$, all the leading principal minors of (61) are positive, while if $|\mu_a| > 1$, not all of them have the same sign. Hence, the matrix is positive definite in the first case and indefinite in the second one.

Thus, from equation (59), the region occupied by this medium is contained in $\Omega_{ep}$ and, from equation (61), it is contained in $\Omega_{mp}$ if $|\mu_a| < 1$. In this case, then, this medium is neither a part of $\Omega_{el}$ nor a part of $\Omega_{dp}$. On the contrary, this medium is contained neither in $\Omega_{mp}$ nor in $\Omega_{mn}$ when $|\mu_a| > 1$.

In any case, independently of the value of $\mu_a \in \mathbb{R} \setminus \{-1, 1\}$, the second square parenthesis of H7 cannot hold true since the region $\Omega_{MK}$ satisfies $\Omega_{MK} \subseteq \Omega_{el} \setminus \Omega_{dl}$ and $\Omega_{MK}$ is never contained in $\Omega_{mn}$.

When $|\mu_a| > 1$ the first square bracket of H7 does not hold true either since, as before $\Omega_{MK} \subseteq \Omega_{el} \setminus \Omega_{dl}$, and $\Omega_{MK}$ is not contained in $\Omega_{mp}$. Thus H7 is violated and our theory cannot be applied.

However, if the bianisotropic medium considered is characterized by $|\mu_a| < 1$, which are the most interesting cases, as pointed out by [35] where $\mu_a$ has been considered belonging to the set $\{-0.25, 0, 0.25\}$, the first square bracket of H7 can still be satisfied.

As a matter of fact, we know that $\Omega_{MK}$ is not contained in $\Omega_{ll}$ so that the presence of such a medium cannot affect the validity of the third parenthesis ($\Omega_{ll} \subseteq \Omega_{Bp}$) of the first square bracket of H7.

Thus, in a model involving the bianisotropic medium considered, lossless materials can be present provided they are magnetic positive and electric negative. If we consider anisotropic media, they would be lossless electric negative (ENG) media. On the contrary, lossless double positive (DPS), magnetic negative (MNG) or double negative (DNG) media cannot be involved together with the bianisotropic medium considered to retain the validity of H7 and the conclusions of our theory.

“Double lossy” medium can be involved without any limitation since $\Omega_{dl}$ is not considered in all parenthesis of H7.

If the model involves, together with the bianisotropic medium considered, an electrically lossy but not magnetically lossy medium, which is then part of $\Omega_{el} \setminus \Omega_{dl}$, in order to retain the validity of H7 and, in particular, of the first parenthesis of the first square bracket of H7, such a medium has to be contained in $\Omega_{mp}$. If, for example, such a medium is anisotropic, it could be a DPS or a ENG medium but it cannot be a DNG or an MNG medium.

Finally, if the model involves, together with the bianisotropic medium considered, a magnetically lossy but not electrically lossy medium, which is then part of $\Omega_{ml} \setminus \Omega_{dl}$, in order to retain the validity of H7 and, in particular, of the second parenthesis of the first square bracket of H7, such a medium has to be contained in $\Omega_{mn}$. If, for example, such a medium is anisotropic, it could be a DNG or an ENG medium but it cannot be a DPS or an MNG medium.
Such considerations apply for example to the radiation and scattering problems, in free space or in waveguides, considered in [6].

Now we provide some numerical results. They can be used to test the implementation of other numerical simulator, since our theory guarantees that the results obtained with a sufficiently fine mesh are reliable.

We consider a waveguide discontinuity problem in which an obstacle made up of a bianisotropic medium is placed in a rectangular waveguide. The length, width and height of the waveguide are, respectively, $d = 12$ cm, $a = 2$ cm and $b = 1$ cm and the working frequency is 10 GHz.

The geometry of the waveguide discontinuity problems considered is reported in figure 1 along with the cartesian frame of reference.

The obstacle in the waveguide is a parallelepiped made up of the medium considered in [35] and discussed above, with $\mu_a = 0$, so that the material is actually biisotropic with $\varepsilon_r = 1.1(1 - j0.01)$, $\mu_r = 1$ and the so-called chirality admittance $\xi = \eta = 0.001$ mS. Its length, width and height are, respectively, 3 cm, 1 cm and 1 cm. It is placed exactly in the middle of the waveguide section considered along the $x$ and $z$ directions.

Apart from the obstacle, the waveguide is filled with air, characterized by $\varepsilon_{\text{air}} = \varepsilon_0 (1 - j\tan(\delta))$ and $\mu_0$; we take account of the fact that, at the frequency of interest, its electric loss tangent is approximately $2 \times 10^{-8}$, so that condition H7 is satisfied (the air region is contained in $\Omega_{el}$ and in $\Omega_{mp}$ but not in $\Omega_{ml}$) for this problem and the results of our theory are guaranteed.

The waveguide walls are considered to be made up of copper and the usual Leontovic boundary conditions at imperfectly conducting surfaces [26] (pp. 384-5) are enforced. On the port at $z = z_1$ the values of $Y$ and $f_R$ appearing in (5) are set to obtain a unit amplitude TE$_{10}$ mode incidence ($\gamma = \alpha + j\beta = \sqrt{\frac{\omega^2}{\sigma^2} - \omega^2 \varepsilon_{\text{air}} \mu_0}$, $\alpha, \beta \in \mathbb{R}^+$, $Y = \frac{\gamma}{j\omega\mu_0}$, $f_R = 2 \sqrt{\frac{2\omega}{ab}} Y \sin \left(\frac{\pi x}{a}\right) e^{-\gamma z_1} \hat{y}$). On the port at $z = z_2 = z_1 + d$ the value of $Y$ is the same as before but $f_R = 0$ so that this port is not excited and the TE$_{10}$ mode is perfectly matched.

The numerical solutions are calculated by using a finite element simulator based on Galerkin’s method [20], [25], with first order edge elements [25] on triangulations of the domain made up of tetrahedra.

The results have been obtained by discretizing the domain with small cubes ($\frac{1}{2n}$ cm on a side, $n \in \mathbb{N}^+$) which in turn are divided into six tetrahedra. In Table 1 we report the different features of the meshes exploited ($w$ represents the maximum side length [cm] of the small cubes in which $\Omega$ is divided).

In Figure 2 we report the magnitude of the $y$ component of the electric field along the line ($x = 1.003$ cm, $y = 0.503$ cm), calculated for the different values of $n$ considered. The expected convergent behaviour of the solution is evident. This fact is confirmed by all the calculated results.
Table 1: Features of the triangulations exploited in the simulations of the waveguide discontinuity problem.

<table>
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<th>n</th>
<th>w</th>
<th>nodes</th>
<th>elements</th>
<th>external faces</th>
<th>edges</th>
</tr>
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<td>608</td>
<td>1830</td>
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<td>9216</td>
<td>2432</td>
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<td>73728</td>
<td>9728</td>
<td>93432</td>
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<td>589824</td>
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<td>2388456</td>
</tr>
</tbody>
</table>

Figure 2: Behaviour of the magnitude of $E_y$ along the line ($x = 1.003 \, \text{cm}, \ y = 0.503 \, \text{cm}$), for different values of $n$. 
Figure 3: Behaviour of the magnitude and of the phase of $E_x$ along the line ($y = 0.503 \text{ cm}, z = 6.03 \text{ cm}$).

Figure 4: Behaviour of the magnitude and of the phase of $E_y$ along the line ($y = 0.503 \text{ cm}, z = 6.03 \text{ cm}$).

As already pointed out, our results can be used to test the reliability of other numerical simulators. For this reason, in Figures 3, 4 and 5 we show the magnitude and the phase of the three cartesian components of the electric field, calculated by using $n = 12$, along the line ($y = 0.503 \text{ cm}, z = 6.03 \text{ cm}$). Analogously, Figures 6, 7 and 8 and Figures 9, 10 and 11 report the same results calculated, respectively, along the line ($x = 1.003 \text{ cm}, z = 6.03 \text{ cm}$) and ($x = 1.003 \text{ cm}, y = 0.503 \text{ cm}$). In the same figures we also report the results calculated when the obstacle is made up of an isotropic medium having the same $P$ and $Q$ as before but ($\xi = \eta = 0$ and then) $L = M = 0$.

As can be seen from figures 3-11, it is important to take into account the chirality admittance, even when its value is small with respect to $|\varepsilon_r|$ and $|\mu_r|$, since the effect of such a constitutive parameter is not reduced by the presence of other factors like $\varepsilon_0$ or $\mu_0$ and is not at all trivial, especially on the electric field depolarization (see, in particular, figures 3, 5, 6 and 9).

7 Conclusions

In this work we have dealt with a boundary value problem for the time harmonic Maxwell system in a rather general setting that permits material configurations involving many bianisotropic
Figure 5: Behaviour of the magnitude and of the phase of $E_z$ along the line ($y = 0.503$ cm, $z = 6.03$ cm).

(a)

(b)

Figure 6: Behaviour of the magnitude and of the phase of $E_x$ along the line ($x = 1.003$ cm, $z = 6.03$ cm).

(a)

(b)

Figure 7: Behaviour of the magnitude and of the phase of $E_y$ along the line ($x = 1.003$ cm, $z = 6.03$ cm).
Figure 8: Behaviour of the magnitude and of the phase of $E_z$ along the line ($x = 1.003$ cm, $z = 6.03$ cm).

Figure 9: Behaviour of the magnitude and of the phase of $E_x$ along the line ($x = 1.003$ cm, $y = 0.503$ cm).

Figure 10: Behaviour of the magnitude and of the phase of $E_y$ along the line ($x = 1.003$ cm, $y = 0.503$ cm).
Figure 11: Behaviour of the magnitude and of the phase of $E_z$ along the line ($x = 1.003$ cm, $y = 0.503$ cm).

materials and metamaterials that fill subregions of the problem domain having Lipschitz continuous boundaries. Sufficient conditions for well posedness and finite element approximability of this problem have been reported. They generalize analogous conditions previously obtained for material configurations involving only anisotropic materials and metamaterials. Even though the aforementioned conditions are only sufficient, many realistic cases are covered by the proposed theory. On the other hand, the cases left out by this theory do not seem amenable to a treatment exploiting the available tools currently used to prove well posedness.

A Appendix: trace operators and Green formulas

In order to define the space $U$ and to recast Problem 1 in variational form, we need trace operators for $H(\text{curl}, \Omega)$ and Green formulas for vector fields in $H(\text{curl}, \Omega)$ and for vector fields in $U$, most of which are developed in [29], [36] and [37]. In the following we briefly recall them, while we refer the reader to the quoted papers for proofs and full details.

Let $\gamma$ and $\pi$ be, as usual, the extensions by continuity to $H(\text{curl}, \Omega)$ of the trace mappings defined in $(C^\infty(\Omega))^3$ [19] (p. 3), [20] (p. 36) by $u \mapsto - u \times \mathbf{n}$ and $u \mapsto - \mathbf{n} \times (u \times \mathbf{n})$, respectively [29].

The following Hilbert spaces are then defined [29]

$$V_{\gamma} = \gamma((H^1(\Omega))^3),$$
$$V_{\pi} = \pi((H^1(\Omega))^3),$$

$$H^{-\frac{1}{2}}(\text{divr}, \Gamma) = \{ a \in V_{\pi}' | \text{divr} a \in H^{-\frac{1}{2}}(\Gamma) \},$$

$$H^{-\frac{1}{2}}(\text{curlr}, \Gamma) = \{ a \in V_{\gamma}' | \text{cirlr} a \in H^{-\frac{1}{2}}(\Gamma) \},$$

where the primes indicate antidual spaces. The corresponding antiduality pairings are denoted $\langle a, b \rangle_{V_{\gamma}'}$ and $\langle a, b \rangle_{V_{\pi}'}$. Both $V_{\gamma}$ and $V_{\pi}$ are dense in $L^2(\Gamma)$ [29] (p. 850).

The nontrivial property that the tangential trace and the tangential component operators

$$\gamma : H(\text{curl}, \Omega) \to H^{-\frac{1}{2}}(\text{divr}, \Gamma)$$

$$\pi : H(\text{curl}, \Omega) \to H^{-\frac{1}{2}}(\text{cirlr}, \Gamma),$$

are linear, continuous, surjective and admit continuous right inverses holds true [29].
The spaces of tangential traces and tangential components \( H^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma) \) and \( H^{-\frac{1}{2}}(\text{curl}_{\Gamma}, \Gamma) \) are antidual each other and if we denote \( H^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma)(a, b) \) the antiduality pairing between them, the Green formula [29]

\[
(u, \text{curl}v)_{0, \Omega} - (\text{curl} u, v)_{0, \Omega} = H^{-\frac{1}{2}}(\text{div}_{\Gamma}, \Gamma) \left( \langle \gamma_{\tau} (u), \pi_{\tau} (v) \rangle \right)_{H^{-\frac{1}{2}}(\text{curl}_{\Gamma}, \Gamma)} \quad \forall u, v \in H(\text{curl}, \Omega) \tag{68}
\]

holds true.

These results have been proved in [29] for bounded, simply connected Lipschitz domains and (as stated in [38]) can be extended to general topology by the results in [36] and [37].

The Green formula [20] (Theorem 3.29)

\[
(u, \text{curl} v)_{0, \Omega} - (\text{curl} u, v)_{0, \Omega} = \langle \gamma_{\tau} (u), v \rangle_{\frac{3}{2}, \Gamma} \quad \forall u \in H(\text{curl}, \Omega) \quad \forall v \in (H^1(\Omega))^3, \tag{69}
\]

where the last term is the antiduality pairing between \((H^{-1/2}(\Gamma))^3\) and \((H^{1/2}(\Gamma))^3\), holds true, too.

Owing to H2, \( n \in (L^\infty(\Gamma))^3 \). Hence, if \( u \in U \), then \( \gamma_{\tau} (u) \in L^2(\Gamma) \) and is just \( u \times n \). With this regularity, the antiduality in (69) reduces to a scalar product in the pivot space \((L^2(\Gamma))^3\) and can be elaborated as follows:

\[
\langle \gamma_{\tau} (u), v \rangle_{\frac{3}{2}, \Gamma} = (u \times n, v)_{0, \Gamma} = \int_{\Gamma} (u \times n) \cdot v \quad \forall u \in H(\text{curl}, \Omega) \quad \forall v \in (H^1(\Omega))^3.
\]

Therefore, we have

\[
(u, \text{curl} v)_{0, \Omega} - (\text{curl} u, v)_{0, \Omega} = (u \times n, n \times v \times n)_{0, \Gamma} \quad \forall u \in U, \quad \forall v \in (H^1(\Omega))^3.
\]

Since \((C^\infty (\Omega))^3 \subset (H^1(\Omega))^3 \subset U\), by [39] or Theorem 3.54 of [20] we easily deduce that \((H^1(\Omega))^3\) is dense in \( U \).

Now, noticing that \( \| n \times v \times n \|_{0, \Gamma} = \| v \times n \|_{0, \Gamma}, \) all the sesquilinear forms involved in (71) are continuous in the \( U \) norm as functions of \( v \). As, moreover, they make sense for \( v \in U \), (71) extends by continuity to

\[
(u, \text{curl} v)_{0, \Omega} - (\text{curl} u, v)_{0, \Omega} = (u \times n, n \times v \times n)_{0, \Gamma} \quad \forall u, v \in U.
\]

Notice that everywhere in this paper, except in the present Appendix, we do not distinguish between \( u \times n \) and \( \gamma_{\tau} (u) \), neither between \( n \times u \times n \) and \( \pi_{\tau} (u) \), and use the cross product notation for both.

**B Appendix: meaning of the proposed definitions**

Some of the subregions of \( \Omega \) involved in condition H7 are defined in terms of the whole \( 6 \times 6 \) matrices \( B \) or \( C \). In order to get a deeper understanding of the meaning of the conditions defining these subregions, let us investigate their implications on the \( 3 \times 3 \) submatrices composing them.

Notice that the diagonal submatrices govern the effects depending either on electric field only or on magnetic field only, while the off-diagonal submatrices govern the effects simultaneously depending on both fields. Hence, this kind of investigation actually shed light on the link between our mathematical definitions and physical properties of materials.

A crucial tool of the aforementioned investigation will be the following lemma.

**Lemma 2.** Let

\[
H = \begin{pmatrix} \alpha & \gamma \\ \gamma^* & \beta \end{pmatrix}
\]

be a \( 6 \times 6 \) hermitian matrix, where \( \alpha, \beta \) and \( \gamma \) are \( 3 \times 3 \) matrices.
Then,
\[ \pm p^* H p \geq K_1|q|^2 + K_2|r|^2 \quad \forall q, r \in \mathbb{C}^3, \] (74)
where \( K_1 \geq 0, K_2 \geq 0 \) and \( p, q \) and \( r \) are linked together by (34), is equivalent to
\[
\begin{cases}
\pm q^* \alpha q \geq K_1|q|^2 & \forall q \in \mathbb{C}^3 \\
\pm r^* \beta r \geq K_2|r|^2 & \forall r \in \mathbb{C}^3 \\
|q^* \gamma r|^2 \leq (\pm q^* \alpha q - K_1|q|^2)(\pm r^* \beta r - K_2|r|^2) & \forall q, r \in \mathbb{C}^3
\end{cases}
\] (75)

**Remark 3.** In Lemma 2 it is understood that we take either all the upper signs or all the lower signs in both (74) and (75), without mixing them together. Double signs in its proof, which follows, are to be interpreted in the same way.

**Proof.** Since \( H \) is hermitian, \( p^* H p \) is a real quantity and (74) makes sense. Moreover, we have \( H^* = H \), which implies \( \alpha^* = \alpha \) and \( \beta^* = \beta \), namely the hermiticity of both \( \alpha \) and \( \beta \). Hence both \( q^* \alpha q \) and \( r^* \beta r \) are real quantities, too, and also (75)\(_1\), (75)\(_2\) and (75)\(_3\) make sense.

By setting \( \hat{q} = \frac{q}{|q|} \) and \( \hat{r} = \frac{r}{r} \), (74), which reads
\[ (\pm q^* \alpha q - K_1)|q|^2 + (\pm r^* \beta r - K_2)|r|^2 \geq 0 \quad \forall q, r \in \mathbb{C}^3, \] (77)
\[ \text{since} \quad (\pm q^* \alpha q - K_1)|q|^2 + (\pm r^* \beta r - K_2)|r|^2 \geq 0 \quad \forall q, r \in \mathbb{C}^3, \] (78)

asking that either (77) or (78) or both are satisfied \( \forall q, r \in \mathbb{C}^3 \) is exactly the same. Asking both is in turn equivalent to
\[ (\pm q^* \alpha q - K_1)x^2 + (\pm r^* \beta r - K_2)y^2 + (\pm q^* \gamma r + r^* \gamma^* q)xy \geq 0 \quad \forall q, r \in \mathbb{C}^3, \quad \forall x, y \in \mathbb{R}, \] (79)
which can be rewritten as
\[ \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \pm q^* \alpha q - K_1 & \pm q^* \gamma r \\ \pm r^* \beta r - K_2 & \pm r^* \gamma^* q \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \geq 0 \quad \forall q, r \in \mathbb{C}^3, \quad \forall x, y \in \mathbb{R}, \] (80)
and, thus, precisely means that the involved \( 2 \times 2 \) hermitian matrix is positive semidefinite \( \forall q, r \in \mathbb{C}^3 \).

Then, since a \( 2 \times 2 \) hermitian matrix is positive semidefinite if and only if its diagonal elements and its determinant are nonnegative, we get
\[
\begin{cases}
\pm q^* \alpha q - K_1 \geq 0 & \forall q \in \mathbb{C}^3 \\
\pm r^* \beta r - K_2 \geq 0 & \forall r \in \mathbb{C}^3 \\
(\pm q^* \alpha q - K_1)(\pm r^* \beta r - K_2) \geq (\pm q^* \gamma r)(\pm r^* \gamma^* q) = |q^* \gamma r|^2 & \forall q, r \in \mathbb{C}^3
\end{cases}
\] (81)
and, finally, by multiplying (81)\(_1\), (81)\(_2\) and (81)\(_3\) by \( |q|^2 \), \( |r|^2 \) and \( |q|^2|r|^2 \), respectively, we obtain the equivalent system (75).

**Remark 4.** Since \( q^* \gamma r + r^* \gamma^* q = (q^* \gamma r + (q^* \gamma r)^*) = 2 \text{Re}(q^* \gamma r) \) for each of the off-diagonal elements of the matrix in (80). Doing so we would prove that also (75) with \( |q^* \gamma r|^2 \) in (75)\(_3\) substituted by \( (\text{Re}(q^* \gamma r))^2 \) is equivalent to (74).

Let us point out that, with suitable choices of \( K_1, K_2 \) and sign, (74) covers all the conditions we are concerned with.

Roughly speaking, Lemma 2 says us that, when (74) is satisfied, the diagonal submatrices have the same (i.e., positive or negative) definiteness of \( H \) and the off-diagonal terms of the quadratic form \( p^* H p \) are “dominated” by the diagonal ones, with a safety margin increasing with the positivity constants \( K_1 \) and \( K_2 \). The following corollaries of Lemma 2 give further details on this by showing what happens in particular cases.
Corollary 1. (75)\textsubscript{1} and (75)\textsubscript{2} are necessary conditions for (74). If $\gamma = 0$, then they are sufficient, too.

Corollary 2. Suppose (74) is satisfied. If $\ker(\alpha) \neq \{0\}$ (respectively, $\ker(\beta) \neq \{0\}$), then $K_1 = 0$ (respectively, $K_2 = 0$) and $\ker(\alpha) \subseteq \ker(\gamma^*)$ (respectively, $\ker(\beta) \subseteq \ker(\gamma)$). In particular, if $\alpha = 0$ (respectively, $\beta = 0$), then $\gamma = 0$ and (74) is equivalent to (75)\textsubscript{2} (respectively, (75)\textsubscript{1}) only.

Proof. Suppose $\exists \tilde{q} \in \ker(\alpha)$, $\tilde{q} \neq 0$. Then, we have $0 = \pm \tilde{q}^* \alpha \tilde{q} \geq K_1 |\tilde{q}|^2$ and $K_1 = 0$ follows. Hence, we have

$$|r^* \gamma^* \tilde{q}|^2 = |(r^* \gamma^* \tilde{q})|^2 = |(\tilde{q}^* \gamma r)^*|^2 \leq 0 \quad \forall r \in \mathbb{C}^3,$$

from which $\gamma^* \tilde{q} = 0$, namely, $\tilde{q} \in \ker(\gamma^*)$. If $\alpha = 0$, then (82) holds $\forall \tilde{q} \in \mathbb{C}^3$. Thus, $\gamma^* = 0$ and, then, $\gamma = 0$. Moreover, (75)\textsubscript{1} and (75)\textsubscript{2} become empty conditions.

Before applying these general statements to our specific conditions, let us introduce also the following lemma.

Lemma 3. (74) holds with $K_1 > 0$ and $K_2 > 0$ if and only if it holds, separately, with $K_1 > 0$ and $K_2 = 0$ and with $K_1 = 0$ and $K_2 > 0$.

Proof. The “only if” part is trivial. The “if” one is as follows: from $\pm p^* H p \geq K_1 |q|^2$ and $\pm p^* H p \geq K_2 |r|^2$ we get $\pm 2p^* H p \geq K_1 |q|^2 + K_2 |r|^2$ and (74) holds true with $K_1 = \frac{K_1}{2}$ and $K_2 = \frac{K_2}{2}$.

Now, we are in a position to justify some statements we gave without proof in Section 4.

First of all, in the case of anisotropic materials (35) is equivalent to (36) and (37) is equivalent to (39) as a trivial consequence of Corollary 1.

Then, Corollary 2 clearly explains why, in the most general case, the magnetic losses cannot vanish in $\Omega_{el}$ and the electric losses cannot vanish in $\Omega_{ml}$. Lemma 3 shows that $\Omega_{el} = \Omega_{el} \cap \Omega_{ml}$, while $\Omega_{B_p} \subset \Omega_{mp} \cap \Omega_{en}$ and $\Omega_{B_n} \subset \Omega_{en} \cap \Omega_{mn}$ are again consequences of Corollary 1.

Let us conclude by some general considerations. Lemma 2 applied to the $C$ matrix shows that, in a passive material, the losses depending on both the electric and the magnetic field are bounded by the purely electric and purely magnetic losses, in the precise sense given by (75)\textsubscript{3}. Moreover, Corollary 2 more explicitly shows that this happens independently for each possible polarization of the fields.

A similar bound does not exist, in general, for the off-diagonal terms of the reactive power because the $B$ matrix is not constrained to be semidefinite since for $B$ there is no counterpart of H5.

However, it arises in both the $B$–positive and the $B$–negative regions because of (45) and (46), respectively.

C Appendix: proof of Theorem 4

As already pointed out in the proof of Theorem 3 the hypotheses H1 and H2 are required to give a precise meaning to the space $U$.

We have

$$a(u, u) = \int_\Omega w^* A w + j \omega \langle Y(n \times u \times n), n \times u \times n \rangle_{0, \Gamma} =$$

$$\left[ \int_\Omega w^* B w - \omega Im \langle Y(n \times u \times n), n \times u \times n \rangle_{0, \Gamma} \right] + j \left[ - \int_\Omega w^* C w + \omega Re \langle Y(n \times u \times n), n \times u \times n \rangle_{0, \Gamma} \right].$$

(83)
As $B$ and $C$ are hermitian matrices, both square brackets are real and, thus, they are, respectively, the real and imaginary part of $a(u, u)$. Hence,

$$
|a(u, u)|^2 = 
\left[ \int_{\Omega} w^* B w - \omega Im \left( Y (n \times u \times n), n \times u \times n \right)_{0, \Gamma} \right]^2
+ \left[ - \int_{\Omega} w^* C w + \omega Re \left( Y (n \times u \times n), n \times u \times n \right)_{0, \Gamma} \right]^2.
$$

(84)

In order to achieve the coercivity of the sesquilinear form, the two square brackets of (84) together must control the $U$ norm. At first, we focus our attention on the second square bracket which takes account of the losses in the media and on the boundary. From $\omega > 0$ and H6 we deduce that $\exists C_1 > 0$ such that

$$
\omega Re \left( Y (n \times u \times n), n \times u \times n \right)_{0, \Gamma} \geq C_1 \| u \times n \|_{0, \Gamma}^2.
$$

(85)

From H5, it follows that $\int_{\Omega_{ll}} w^* C w \leq 0$. Then, by using the definition of $\Omega_{ll}$ and the additivity of integrals, we get

$$
\int_{\Omega} w^* C w
= \int_{\Omega_{ll}} w^* C w + \int_{\Omega_{di} \cup \Omega_{ml} \cup \Omega_{df}} w^* C w
\leq \int_{\Omega_{di} \cup \Omega_{ml} \cup \Omega_{df}} w^* C w
= \int_{\Omega_{di}} w^* C w + \int_{\Omega_{ml} \setminus \Omega_{di}} w^* C w + \int_{\Omega_{df}} w^* C w
\leq -K_{el} \| u \|_{0, \Omega_{di} \setminus \Omega_{df}}^2 - K_{ml} \| \text{curl} u \|_{0, \Omega_{ml} \setminus \Omega_{di}}^2 - K_{df} \| n \times u \|_{0, \Omega_{df}}^2.
$$

(86)

Hence, we have

$$
- \int_{\Omega} w^* C w \geq \min \{ K_{di}, K_{df} \} \| u \|_{0, \Omega_{di} \cup \Omega_{df}}^2 + \min \{ K_{ml}, K_{df} \} \| \text{curl} u \|_{0, \Omega_{ml} \cup \Omega_{df}}^2.
$$

(87)

By posing $C_2 = \min \{ K_{di}, K_{ml}, K_{df}, C_1 \}$ and adding together (85) and (87) we obtain

$$
- \int_{\Omega} w^* C w + \omega Re \left( Y (n \times u \times n), n \times u \times n \right)_{0, \Gamma}
\geq C_2 \left( \| u \|_{0, \Omega_{di} \cup \Omega_{df}}^2 + \| \text{curl} u \|_{0, \Omega_{ml} \cup \Omega_{df}}^2 + \| n \times u \|_{0, \Gamma}^2 \right).
$$

(88)

Now that we have achieved part of the control we need by the second square bracket of (84), we switch to the first one. We are going to split the terms involved since, where we have a positive contribution in inequality (88), we can admit moderately negative contributions from the reactive part without disrupting the already achieved control. For this reason we define

$$
s_{a1} = c \left( \frac{Q + Q^*}{2} \text{curl} u, \text{curl} u \right)_{0, \Omega_{ml} \cup \Omega_{df}}
$$

(89)

$$
s_{a2} = c \left( \frac{Q + Q^*}{2} \text{curl} u, \text{curl} u \right)_{0, \Omega_{di}}
$$

(90)

$$
s_{a3} = c \left( \frac{Q + Q^*}{2} \text{curl} u, \text{curl} u \right)_{0, \Omega_{di} \setminus \Omega_{df}}
$$

(91)

$$
s_{b1} = \frac{\omega^2}{c} \left( \frac{P + P^*}{2} u, u \right)_{0, \Omega_{di} \setminus \Omega_{df}}
$$

(92)
Hence, the first square bracket of (84) becomes

\[
\int_\Omega w^* B w - \omega Im(Y(n \times u \times n), n \times u \times n)_{0, \Gamma} \leq (s_{a1} + s_{a2} + s_{a3} - s_{b1} - s_{b2} - s_{b3} +
\]

\[
\quad \quad \quad s_{c1} + s_{c2} + s_{c3} - \omega Im(Y(n \times u \times n), n \times u \times n)_{0, \Gamma})^2
\]

\[
\leq (s_{a2} + s_{a3} - s_{b2} - s_{b3} + s_{c2} + s_{c3}) +
\]

\[
\quad \quad \quad - (s_{a1} + s_{b1} - s_{c1} + \omega Im(Y(n \times u \times n), n \times u \times n)_{0, \Gamma})^2
\]

\[
\geq (1 - \alpha)(s_{a2} + s_{a3} - s_{b2} - s_{b3} + s_{c2} + s_{c3})^2 +
\]

\[
\quad \quad \quad (1 - \frac{1}{\alpha}) (-s_{a1} + s_{b1} - s_{c1} + \omega Im(Y(n \times u \times n), n \times u \times n)_{0, \Gamma})^2
\]

for all $\alpha \in \mathbb{R}$, $\alpha > 0$, having exploited the formula $2ab \leq \alpha a^2 + \frac{1}{\alpha} b^2$.

We have to control the amplitudes of the terms of the last parenthesis since they will provide a negative contribution as a consequence of the fact that $\alpha$ will be chosen < 1 so that $(1 - \frac{1}{\alpha}) < 0$. Owing to H3, there exist three positive constants $C_3$, $C_4$ and $C_5$ such that

\[
|s_{a1}| \leq C_3\|\text{curl } u\|_{0, \Omega_{\text{el}}}^2
\]

\[
|s_{b1}| \leq C_4\|u\|_{0, \Omega_{\text{el}}}^2
\]

\[
|s_{c1}| \leq C_5\|u\|_{0, \Omega_{\text{el}}}\|\text{curl } u\|_{0, \Omega_{\text{el}}}.
\]

From $\omega > 0$ and H4 it follows that $\exists C_6 > 0$ such that

\[
| - \omega Im(Y(n \times u \times n), n \times u \times n)_{0, \Gamma}| \leq \omega C_6\|n \times u \times n\|_{0, \Gamma}^2.
\]

Then, from (102), (99), (100) and (101) we get

\[
| - s_{a1} + s_{b1} - s_{c1} + \omega Im(Y(n \times u \times n), n \times u \times n)_{0, \Gamma}|
\]

\[
\leq |s_{a1}| + |s_{b1}| + |s_{c1}| + \omega |Im(Y(n \times u \times n), n \times u \times n)_{0, \Gamma}|
\]

\[
\leq C_3\|\text{curl } u\|_{0, \Omega_{\text{el}}}^2 + C_4\|u\|_{0, \Omega_{\text{el}}}^2 + C_5\|u\|_{0, \Omega_{\text{el}}}\|\text{curl } u\|_{0, \Omega_{\text{el}}} + \omega C_6\|n \times u \times n\|_{0, \Gamma}^2.
\]

Thus $\exists C_7 > 0$ such that

\[
\left[ - s_{a1} + s_{b1} - s_{c1} + \omega Im(Y(n \times u \times n), n \times u \times n)_{0, \Gamma}\right]^2
\]

\[
\leq C_7(\|\text{curl } u\|_{0, \Omega_{\text{el}}}^2 + \|u\|_{0, \Omega_{\text{el}}}^2 + \|u\|_{0, \Omega_{\text{el}}}\|\text{curl } u\|_{0, \Omega_{\text{el}}} + \|n \times u \times n\|_{0, \Gamma}^2).
\]
Since \( ab \leq (a^2/2) + (b^2/2) \) \( \forall (a, b) \in \mathbb{R}^2 \), then

\[
\left[ -s_{a1} + s_{b1} - s_{c1} + \omega Im(Y(n \times u \times n), n \times u \times n)_{0,1} \right]^2 \\
\leq C_{r}^2 \left( \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \frac{1}{2} \left\| u \right\|_{0, \Omega_{dt}}^2 + \frac{1}{2} \left\| \text{curl} u \right\|_{0, \Omega_{dt}}^2 + \left\| n \times u \times n \right\|_{0,1}^2 \right)^2.
\]  
(105)

Therefore \( \exists C_{s} > 0 \) such that

\[
\left[ -s_{a1} + s_{b1} - s_{c1} + \omega Im(Y(n \times u \times n), n \times u \times n)_{0,1} \right]^2 \\
\leq C_{s}^2 \left( \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| n \times u \times n \right\|_{0,1}^2 \right)^2.
\]  
(106)

For any \( \alpha \) such that \( 0 < \alpha < 1 \) we have \( 1 - \alpha > 0 \) and \( (1 - (1/\alpha)) < 0 \). Then, we deduce that \( \forall \alpha \in \mathbb{R}, 0 < \alpha < 1 \) we have

\[
\left\| a(u, u) \right\|^2 \\
\geq (1 - \alpha)(s_{a2} + s_{a3} - s_{b2} - s_{b3} + s_{c2} + s_{c3})^2 \\
+ (1 - \frac{1}{\alpha})C_{s}^2 \left( \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| n \times u \times n \right\|_{0,1}^2 \right)^2 \\
+ C_{s}^2 \left( \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| n \times u \times n \right\|_{0,1}^2 \right)^2 \\
= (1 - \alpha)(s_{a2} + s_{a3} - s_{b2} - s_{b3} + s_{c2} + s_{c3})^2 \\
+ \left( C_{s}^2 + (1 - \frac{1}{\alpha})C_{s}^2 \right) \left( \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| n \times u \times n \right\|_{0,1}^2 \right)^2.
\]  
(107)

By choosing \( 1 > \alpha > \frac{C_{s}^2}{C_{s}^2 + C_{s}^2} > 0 \) then both \( 1 - \alpha \) and \( C_{s}^2 + (1 - \frac{1}{\alpha})C_{s}^2 \) are positive.

Now we switch our attention to the first addend of the last term. We need a positive contribution from this part. Owing to H3, we have

\[
\left| s_{c3} \right| \leq C_{9} \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}} \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}} \left\| n \times u \times n \right\|_{0,1}.
\]  
(108)

for some positive constants \( C_{9} \).

By exploiting the above estimates and the inequalities

\[
\sqrt{2(a^2 + b^2)} \geq |a| + |b| \geq |a + b| \geq |a| - |b|,
\]  
(109)

we can recast (107) as

\[
\sqrt{2} \left| a(u, u) \right| \\
\geq \sqrt{1 - \alpha} \left| s_{a2} + s_{a3} - s_{b2} - s_{b3} + s_{c2} + s_{c3} \right| \\
+ \sqrt{C_{s}^2 + (1 - \frac{1}{\alpha})C_{s}^2} \left( \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| n \times u \times n \right\|_{0,1}^2 \right) \\
\geq \sqrt{1 - \alpha} \left( \left| s_{a2} + s_{a3} - s_{b2} + s_{c2} + s_{a3} - s_{b3} \right| - \left| s_{c3} \right| \right) \\
+ \sqrt{C_{s}^2 + (1 - \frac{1}{\alpha})C_{s}^2} \left( \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| n \times u \times n \right\|_{0,1}^2 \right) \\
\geq \sqrt{1 - \alpha} \left| s_{a2} - s_{b2} + s_{c2} + s_{a3} - s_{b3} \right| - \sqrt{1 - \alpha} C_{9} \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}} \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}} \\
- \sqrt{1 - \alpha} C_{9} \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}} \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}} \\
+ \sqrt{C_{s}^2 + (1 - \frac{1}{\alpha})C_{s}^2} \left( \left\| \text{curl} u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| u \right\|_{0, \Omega_{m1} \cup \Omega_{dt}}^2 + \left\| n \times u \times n \right\|_{0,1}^2 \right).
\]  
(110)
Hence, by setting
\[
(s_{a2} - s_{b2} + s_{c2}) = \int_{\Omega_d} w^* B w,
\]
it is easily verified that, when the first term of the disjunction in H7 is true, we have
\[
\left| (s_{a2} - s_{b2} + s_{c2}) + s_{a3} - s_{b3} \right| 
\geq K_{Bp} (\|u\|_{0,\Omega_d}^2 + \|\text{curl } u\|^2_{0,\Omega_d^c}) + K_{mp} \|\text{curl } u\|^2_{0,\Omega_m^c} \Omega_d + K_{en} \|u\|^2_{0,\Omega_m^c} \Omega_d^c,
\]
while, when the second term is true, the same inequality holds with $K_{Bp}$, $K_{mp}$ and $K_{en}$ substituted by $K_{Bn}$, $K_{mn}$ and $K_{ep}$, respectively. Hence, the proof is the same in both cases and we consider only the first one, in which we obtain
\[
\sqrt{2} \|a(u, u)\| 
\geq \sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} (\|u\|^2_{0,\Omega_d} + \|\text{curl } u\|^2_{0,\Omega_d^c} + \|n \times u \times n\|^2_{0,\Gamma}) 
+ K_{Bp} \sqrt{1 - \alpha} (\|u\|^2_{0,\Omega_d} + \|\text{curl } u\|^2_{0,\Omega_d^c}) 
+ K_{en} \sqrt{1 - \alpha} \|\text{curl } u\|^2_{0,\Omega_m^c} \Omega_d + \sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} \|\text{curl } u\|^2_{0,\Omega_m^c} \Omega_d^c 
- C_9 \sqrt{1 - \alpha} \|u\|^2_{0,\Omega_m^c} \Omega_d + \sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} \|u\|^2_{0,\Omega_m^c} \Omega_d^c 
- C_9 \sqrt{1 - \alpha} \|\text{curl } u\|^2_{0,\Omega_m^c} \Omega_d^c + \sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} \|\text{curl } u\|^2_{0,\Omega_m^c} \Omega_d^c.
\]
The sum of the terms concerning the subregion $\Omega_m^c \setminus \Omega_d^c$ can be regarded as the quadratic form
\[
(x \ y) \left( \sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} - \frac{C_9}{2} \sqrt{1 - \alpha} \right) \begin{pmatrix} x \\ y \end{pmatrix}
\]
evaluated in $x = \|\text{curl } u\|_{0,\Omega_m^c} \Omega_d^c$, $y = \|u\|_{0,\Omega_m^c} \Omega_d^c$. Since we already assumed that $0 < \frac{C^2_2}{C^2_2 + C^2_8} \leq \alpha < 1$, both the diagonal coefficients are positive. Thus, the quadratic form will be positive definite if its determinant
\[
\Delta = K_{en} \sqrt{1 - \alpha} \sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} - \frac{C_9^2}{4} (1 - \alpha)
\]
is positive, namely if
\[
\sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} > \frac{C_9^2}{4K_{en}} \sqrt{1 - \alpha}.
\]
Dealing in the same way with the terms involving $\Omega_d^c \setminus \Omega_m^c$ leads to the similar condition
\[
\sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} > \frac{C_9^2}{4K_{mp}} \sqrt{1 - \alpha}.
\]
Now, notice that $\forall K > 0 \exists \alpha \in (\frac{C^2_2}{C^2_2 + C^2_8}, 1)$ such that $\sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} > K \sqrt{1 - \alpha}$. This happens because, when $\alpha \rightarrow 1^-$, the left-hand side tends to $C_2 > 0$, while the right-hand side vanishes. Hence, by setting $K = \max\{\frac{C^2_2}{C^2_2 + C^2_8}, \frac{C^2_2}{C^2_2 + C^2_8}\}$ and choosing $\alpha$ such that the latter inequality is fulfilled, we can always make simultaneously positive definite both the quadratic forms. Denoting $C_{10} > 0$ and $C_{11} > 0$, respectively, the minimum eigenvalue of the first and of the second quadratic form, we then get
\[
\sqrt{2} \|a(u, u)\| 
\geq \sqrt{C^2_2 + (1 - \frac{1}{\alpha})C^2_8} (\|u\|^2_{0,\Omega_d} + \|\text{curl } u\|^2_{0,\Omega_d^c} + \|n \times u \times n\|^2_{0,\Gamma})
\[ +K_{B_p}\sqrt{1-\alpha} \left( \|u\|_{0,\Omega_i}^2 + \|\text{curl } u\|_{0,\Omega_i}^2 \right) \]
\[ +C_{10} \left( \|u\|_{0,\Omega_{ml}}^2 \right) \Omega_i + \|\text{curl } u\|_{0,\Omega_{ml}}^2 \right) \]
\[ +C_{11} \left( \|u\|_{0,\Omega_{el}}^2 \right) \Omega_i + \|\text{curl } u\|_{0,\Omega_{el}}^2 \right) \). \]

Since all the coefficients in the above inequality are positive, by setting

\[ C_{12} = \min\left\{ \sqrt{C_2^2 + (1 - \frac{1}{\alpha})C_8^2}, K_{B_p}\sqrt{1-\alpha}, C_{10}, C_{11} \right\} > 0 \] (116)

we finally obtain

\[ |a(u, u)| \geq \frac{C_{12}}{\sqrt{2}} \left( \|u\|_{0,\Omega}^2 + \|\text{curl } u\|_{0,\Omega}^2 + \|n \times u \times n\|_{0,\Gamma}^2 \right). \] (117)

References


