Counterexamples to the currently accepted explanation for spurious modes and necessary and sufficient conditions to avoid them

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Abstract

Recently a new theory, concerning the spurious free approximation of electromagnetic eigenproblems, has appeared, which is in contrast with the one currently accepted in the electromagnetic community. To stress the need to adopt the new theory we show that in two simple cases spurious modes appear even though the conditions on which the old and accepted theory is based are satisfied. On the contrary, none of these cases conflicts with the new theory. A new and more subtle behavior of spurious modes is also pointed out.

Keywords: Cavity resonators; finite element methods; modeling; spectral analysis; waveguides.

1 Introduction

Spurious modes appearing in the finite element (FE) solution of electromagnetic eigenproblems have been an object of research for many years [1, 2]. In this long period some theories trying to explain this phenomenon, have been proposed [3–6] and some of them have then been disproved [7–10]. One of these theories [4–6] has been widely accepted in the electromagnetic community and is now regarded as the conclusive explanation of the spurious-mode phenomenon [10, 11].

Recently, the present authors proposed a new theory [12], which is in contrast with the accepted one [12–14], and exploited it to deduce general results of interest in the applications [14, 15]. Nevertheless, the electromagnetic community seems insufficiently aware of this progress, yet.

In this paper, to convince the reader that the currently accepted theory is no longer tenable, we present two counterexamples to it. In particular, we show that lagrangian elements on criss-cross meshes should be spurious-free according to the current theory, but actually are not. Then we show the same for the $Q_{11}$ quadrilateral edge elements [10]. In both cases, the occurrence of spurious modes is compatible with the new theory. Moreover, in the case of the $Q_{11}$ elements, we also prove that a condition which our theory indicates as necessary for the absence of spurious modes is violated.

2 Problem formulation and its approximation

Let us consider a resonant cavity, filled with a linear, homogeneous and loss-free medium. Denote by $\Omega$ the simply connected space region inside the cavity and assume that $\Omega$ is a Lipschitz polyhedron and that its simply connected boundary surface $\Gamma$ is made of a perfect electric conductor (see [12, 15] for a much more general setting).

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The problem of finding its resonant modes can be written in variational form as follows:

**Problem 1.** Find \( k \in \mathbb{R} \) and \( \vec{E} \in V, \vec{E} \neq 0 \), such that \( (\nabla \times \vec{E}, \nabla \times \vec{w}) = k^2 (\vec{E}, \vec{w}) \) \( \forall \vec{w} \in V \),

where \((\vec{u}, \vec{v}) = \int_{\Omega} \vec{u} \cdot \vec{v} \, d\Omega \) is the scalar product in \( H = L^2(\Omega)^3 \) and \( V = H_0(\text{curl}, \Omega) = \{ \vec{v} \mid \vec{v} \in H, \vec{n} \times \vec{v} \mid_{\Gamma} = 0 \} \). \( V \) is endowed with the scalar product \((\vec{u}, \vec{v})_V = (\vec{u}, \vec{v}) + (\nabla \times \vec{u}, \nabla \times \vec{v})\).

The orthogonal decomposition \( V = V_0 \oplus V_1 \) holds true, where \( \oplus \) denotes the direct sum and \( V_0 = \{ \vec{v} \mid \vec{v} \in V, \nabla \times \vec{v} = 0 \} \) is orthogonal in \( V \) to \( V_1 = \{ \vec{v} \mid \vec{v} \in V \cap W, \nabla \cdot \vec{v} = 0 \} \), with \( W = H(\text{div}, \Omega) = \{ \vec{v} \mid \vec{v} \in H, \nabla \cdot \vec{v} \in L^2(\Omega) \} \). The norms corresponding to \((, )\) and to \((, )_V\) will be denoted by \( \| \cdot \|_H \) and \( \| \cdot \|_V \), respectively.

A Galerkin FE approximation of Problem 1 is:

**Problem 2.** Find \( k_h \in \mathbb{R} \) and \( \vec{E}_h \in V_h, \vec{E}_h \neq 0 \), such that \( (\nabla \times \vec{E}_h, \nabla \times \vec{w}_h) = k_h^2 (\vec{E}_h, \vec{w}_h) \) \( \forall \vec{w}_h \in V_h \),

where \( V_h \) is a finite dimensional subspace of \( V \) spanned by a FE basis, \( h \in I \) is the mesh parameter and \( I \) is a denumerable and bounded set of strictly positive indexes accumulating only at zero. In an analogous way we have \( V_h = V_{0h} \oplus V_{1h} \), where \( V_{0h} = V_h \cap V_0 \) and \( V_{1h} \not\subset V_1 \) is its orthogonal complement in \( V_h \).

### 3 Different conditions and theories

For any FE approximation of a problem having solution in \( V \), it is classical to require that the sequence of FE spaces \( \{ V_h \}_{h \in I} \) generated on a “regular family of triangulations” [16] satisfies the condition

\[
\forall \vec{v} \in V \lim_{h \to 0} \inf_{\vec{v}_h \in V_h} \| \vec{v} - \vec{v}_h \|_V = 0. \tag{1}
\]

Three other conditions are relevant in this context:

\[
\forall \vec{v} \in V_0 \lim_{h \to 0} \inf_{\vec{v}_h \in V_{0h}} \| \vec{v} - \vec{v}_h \|_V = 0, \tag{2}
\]

“Any sequence \( \{ \vec{v}_h \}_{h \in I} \) s. t. \( \vec{v}_h \in V_{1h}, \| \vec{v}_h \|_V \leq C \), contains a sub-sequence converging in \( H \), i.e.,

\[
\exists J \subset I \exists \vec{v} \in H \text{ s. t. } \lim_{h \to 0} \| \vec{v} - \vec{v}_h \|_H = 0
\]

\( \forall h \in I \) \( V_h \) contains all the gradients of the first order lagrangian elements on the same mesh”. \( \tag{3} \)

It can be shown that (4) implies (2).

The currently accepted theory can be stated as follows:

**Proposition 1.** Any FE method satisfying (1) and (2) is spurious-free.

Such a statement can be found in [5] and (in an implicit way) in [6]. In fact, (2) is actually the crucial property upon which the proof of Proposition 1 of [6] relies, in spite of the fact that the property explicitly mentioned in [6] is (4), instead. Edge elements fulfill (2) because they fulfill (4), but this fact is not essential since the same proof actually applies to any element just satisfying (2). Nevertheless, many colleagues would be more satisfied with a variant of Proposition 1 in which (2) were substituted by (4). Hence, also this variant will be considered.

Our theory [12–14] reads as follows:

**Proposition 2.** The FE approximation of Problem 1 defined by Problem 2 is spurious-free if and only if the mutually independent conditions (1), (2) and (3) are satisfied.

Proposition 1 and 2 are conflicting, of course. However, the behaviour of vector lagrangian elements on general meshes, which are affected by spurious modes and satisfy only (1), and the behaviour of most of edge elements, which are spurious-free and satisfy (1), (2), (3) and (4) [14,15], are compatible with both theories. In the next section, we exhibit examples permitting to discriminate between these conflicting theories.
4 Counterexamples to the accepted theory

The arguments of Sections 2 and 3 apply, with obvious minor changes, also to the analogous 2D model of a hollow waveguide for the determination of the transverse electric field of TE modes at cutoff.

The first counterexample concerns a FE approximation of this model that satisfies the assumptions of Proposition 1 and nevertheless is corrupted by spurious modes, as mathematically proved and numerically confirmed.

Exactly the same things can be said of the second counterexample, but in this case (2) is satisfied because (4) is satisfied. Hence, it disproves also the mentioned variant of Proposition 1. Moreover, we prove also that (3) is not satisfied, in this case. Therefore, while the presence of spurious modes disproves the old theory (i.e. the preferred version of Proposition 1), the new one (i.e. Proposition 2) says that spurious modes must necessarily occur.

Remark 1. Since Proposition 2 is not a conjecture, but has been rigorously proved in [12], we deduce that (3) is violated also in the first counterexample, even though we have not developed a direct proof of that.

4.1 Lagrangian elements on criss-cross meshes

In this section, let \( \Omega \) be a square domain. Suppose we discretize the variational problem by first order Lagrangian elements on a family of criss-cross meshes (i.e. obtained from a mesh of square elements by further subdividing each square in four equal triangles, by drawing its diagonals). It has been numerically shown in [17] and mathematically proved in [18] (Theorems 5.1 and 5.2) that this approximation is affected by spurious modes.

In [18], actually, it is the \( \text{div} \) operator that plays the role that in our setting is played by the \( \text{curl} \) operator. In 2D, however, \( H(\text{div}, \Omega) \) and \( H(\text{curl}, \Omega) \) are linked by an isometric isomorphism (the \( \mathcal{R} \) operator defined in the proof of Proposition 4 below). Hence, each result in a setting has a counterpart in the other one (e.g. (5) and (6) in the same proof).

By showing that the above spurious affected approximation satisfies both (1) and (2), we prove that Proposition 1 is false.

Proposition 3. First order lagrangian elements on a family of criss-cross meshes satisfy (1).

Proof. Let \( \Psi = H^1(\Omega) = \{ \psi \in L^2(\Omega), \nabla \psi \in L^2(\Omega)^2 \} \) and \( \Phi = H^1_0(\Omega) = \{ \varphi | \varphi \in H^1(\Omega), \varphi|_{\partial \Omega} = 0 \} \). \( \Psi_0 \subset \Psi \) is generated on a criss-cross mesh by scalar first order Lagrangian elements and \( \Phi_0 = \Psi_0 \cap \Phi \). As a family of criss-cross triangulations is regular, standard interpolation results [16] entail that \( \lim_{h \to 0} \inf_{\varphi \in \Phi_0} \| \varphi - \psi_h \|_\Phi = 0 \) \( \forall \varphi \in \Phi \). Hence, for (vector) first order Lagrangian elements we obtain \( \lim_{h \to 0} \inf_{\psi \in \Phi_0^d \cap V_0} \| \psi - \psi_h \|_\Phi = 0 \) \( \forall \psi \in \Phi^d = \Phi \times \Phi \), from which \( \lim_{h \to 0} \inf_{\psi \in \Phi_0^d} \| \psi - \psi_h \|_V = 0 \) \( \forall \psi \in V = H_0(\text{curl}, \Omega) \) follows since \( \Phi^d \) is continuously embedded in \( V \) with dense image. Set \( V_h = \Psi^d_0 \cap V \) and, as \( \Phi^d_0 \subset V_h \), (1) immediately follows. \( \square \)

Proposition 4. First order lagrangian elements on a family of criss-cross meshes satisfy (2).

Proof. From Theorem 3.10 of [18] easily follows that the FE space \( V_h \) of the first order Lagrangian elements on a criss-cross mesh satisfy, for all \( h \in I \), the condition

\[
\inf_{\bar{w}_h \in V_h, \nabla \cdot \bar{w}_h \neq 0} \sup_{0 \neq \bar{z}_h \in V_h} \frac{\langle \nabla \cdot \bar{z}_h, \nabla \cdot \bar{w}_h \rangle_{H(\text{div}, \Omega)}}{\|\bar{z}_h\|_{H(\text{div}, \Omega)} \|\bar{w}_h\|_{L^2(\Omega)}} \geq \beta > 0.
\]

(5)

Given a vector \( \vec{w} = w_x \vec{e}_x + w_y \vec{e}_y \), define the operator \( R \) such that \( \vec{u} = R\vec{w} = -w_y \vec{e}_x + w_x \vec{e}_y \). It is easy to check that \( \langle \nabla \cdot \vec{u} \rangle \vec{e}_z = \nabla \times \vec{u} \), \( \|\vec{u}\|_{H(\text{div}, \Omega)} = \|\vec{u}\|_V \) and \( R(V_h) = V_h \). By setting \( \vec{u}_h = R\vec{w}_h \) and \( \vec{v}_h = R\vec{z}_h \), condition (5) becomes

\[
\inf_{\vec{u}_h \in V_h, \nabla \times \vec{u}_h \neq 0} \sup_{0 \neq \vec{v}_h \in V_h} \frac{\langle \nabla \times \vec{v}_h, \nabla \times \vec{u}_h \rangle_{H}}{\|\vec{v}_h\|_V \|\nabla \times \vec{u}_h\|_H} \geq \beta > 0.
\]

(6)
As the left-hand side is equivalent to
\[
\inf_{0 \neq \tilde{u}_h \in V_h} \left\{ \frac{1}{\|
abla \times \tilde{u}_h\|_H} \sup_{0 \neq \tilde{v}_h \in V_h} \frac{(\nabla \times \tilde{v}_h, \nabla \times \tilde{u}_h)}{\|	ilde{v}_h\|_V} \right\}
\] (7)
and to
\[
\inf_{0 \neq \tilde{u}_h \in V_h} \left\{ \frac{1}{\|
abla \times \tilde{u}_h\|_H} \frac{(\nabla \times \tilde{u}_h, \nabla \times \tilde{u}_h)}{\|	ilde{u}_h\|_V} \right\},
\] (8)
we obtain
\[
\inf_{0 \neq \tilde{u}_h \in V_h} \left\{ \frac{\|
abla \times \tilde{u}_h\|_H}{\|	ilde{u}_h\|_V} \right\} \geq \beta > 0, \forall h \in I,
\] (9)
which is equivalent to
\[
\|
abla \times \tilde{u}_h\|_H \geq \beta \|	ilde{u}_h\|_V, \forall \tilde{u}_h \in V_h \quad \text{and} \quad \forall h \in I.
\] (10)
Then, the application of Proposition 2.21 of [12], which states that (1) and (10) together imply (2), completes the proof.

\[\square\]

**Remark 2.** In this counterexample (4) is not satisfied. Hence, it does not disprove the variant of Proposition 1.

### 4.2 Edge elements on meshes made up of rectangles

In this section, let \(\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq W, 0 \leq y \leq H\} \). Suppose we discretize the variational problem by using quadrilateral edge elements \(Q_{1,1} \) [10] (corresponding to the two-dimensional version of Nédélec’s elements defined on cubes [19]) and a family of uniform meshes made up of \(m\) by \(m\) rectangular elements. Suppose, moreover, that \(m = 2^i, i \in \mathbb{N}\). The following properties are well known [10, 19]:

**Proposition 5.** The just defined FE space sequence satisfies (1), (4) and, then, (2).

It is known from practical experience that \(Q_{1,1}\) elements produce spurious modes [10]. Here, we prove this fact by providing the analytical expression of sequences of spurious modes that, moreover, violate (3).

If we consider a generic rectangular element \(K = \{(x, y) \in \mathbb{R}^2 \mid y_1 \leq y \leq y_2, x_1 \leq x \leq x_2\}\) (with \(y_1 < y_2\) and \(x_1 < x_2\)) the basis functions defining \(Q_{1,1}\) can be defined as follows (see [10], where a different but equivalent set of basis functions is reported):

\[
\begin{align*}
\tilde{w}_1 &= \tilde{e}_x, \\
\tilde{w}_2 &= \frac{y - y_2}{y_1 - y_2}, \\
\tilde{w}_3 &= \frac{y - y_1}{y_2 - y_1}, \\
\tilde{w}_4 &= \frac{2x - x_1 - x_2}{x_2 - x_1}, \\
\tilde{w}_5 &= \frac{x - x_2}{x_1 - x_2}, \\
\tilde{w}_6 &= \frac{2y - y_1 - y_2}{y_2 - y_1}, \\
\tilde{w}_7 &= \frac{x - x_1}{x_2 - x_1}, \\
\tilde{w}_8 &= \frac{2y - y_1 - y_2}{y_2 - y_1}.
\end{align*}
\]

where \(\tilde{e}_x\) and \(\tilde{e}_y\) are the coordinate unit vectors. Note that each basis function has a nonzero tangential component along one edge only: \(\tilde{w}_1\) and \(\tilde{w}_2\) along the edge \(y = y_1\), \(\tilde{w}_3\) and \(\tilde{w}_4\) along \(y = y_2\), \(\tilde{w}_5\) and \(\tilde{w}_6\) along \(x = x_1\) and \(\tilde{w}_7\) and \(\tilde{w}_8\) along \(x = x_2\).

Note also that the two basis functions associated to the same edge interpolate the tangential component with different polynomial orders. This means that in order to define a \emph{curl} conforming approximation each basis function must be matched with only the corresponding basis function of the adjacent element. Then, in particular, we can write \(V_h = X_h \oplus W_h\) (see [15] for analogous
decompositions of $V_h$, where $X_h \subset V$ is the FE space generated by the basis functions $\vec{w}_1, \vec{w}_3, \vec{w}_5$ and $\vec{w}_7$ and $W_h \subset V$ is the FE space generated by the basis functions $\vec{w}_2, \vec{w}_4, \vec{w}_6$ and $\vec{w}_8$.

By direct inspection of the expressions of the basis functions we can see that $X_h$ and $W_h$ are orthogonal in both $H$ and $V$, since both orthogonalities hold true on every element of the mesh.

The space $W_h$ can be further decomposed into two subspaces $Y_h \subset V$ and $Z_h \subset V$, again orthogonal in both $H$ and $V$. $Y_h$ is generated by $\vec{w}_2, \vec{w}_4$ and $Z_h$ is generated by $\vec{w}_6$ and $\vec{w}_8$.

Thus, $V_h = X_h \oplus Y_h \oplus Z_h$, with all subspaces orthogonal to one another. Let us denote by $X_{0h}$ the space $X_h \cap V_0$ and by $X_{1h}$ its orthogonal complement in $X_h$. Analogously, we define $Y_{0h}, Y_{1h}, Z_{0h}$ and $Z_{1h}$.

**Proposition 6.** Any eigenpair $(k_h, \vec{v}_h)$ of Problem 2 with the FE space restricted to $X_h$ or $Y_h$ or $Z_h$ is an eigenpair of Problem 2.

**Proof.** Let us consider the $X_h$ case only, the others being identical. Suppose $(k_h, \vec{v}_h) \in \mathbb{R} \times X_h$ satisfies $(\nabla \times \vec{v}_h, \nabla \times \vec{x}_h) = k_h^2 (\vec{v}_h, \vec{x}_h) \forall \vec{x}_h \in X_h$. Since the previously deduced orthogonalities imply that $(\nabla \times \vec{v}_h, \nabla \times \vec{w}_h) = (\vec{v}_h, \vec{w}_h) = 0 \forall \vec{w}_h \in X_h \forall \vec{v}_h \in Y_h \oplus Z_h$, it can be seen by adding $\vec{w}_h$ to $\vec{x}_h$ that $(k_h, \vec{v}_h)$ satisfies Problem 2.

Thanks to the above Proposition it makes sense to analyise, for example, the properties of the eigenpairs $(k_h, \vec{v}_h) \in \mathbb{R} \times X_h$. Before going on, let $S_i^m, i = 1, \ldots, m$, be the vertical strip (made up of $m$ elements) $S_i^m = \{(x,y) \in \mathbb{R}^2 : x_i^m \leq x \leq x_{i+1}^m, 0 \leq y \leq H\}$, where $x_i^m = (i-1)W/m$, $i = 1, \ldots, m+1$, and let $Y_{h,S_i}$ be the space of vector fields generated by the restrictions to $S_i^m$ of all vector fields belonging to $Y_h$ and let $L^2_D$ be the space of the functions $g(y)$, vanishing at $y = 0$ and $y = H$, generated by first order one-dimensional lagrangian elements on the discretization of $[0, H]$ induced by the nodes $y_i^m = (i-1)H/m, i = 1, \ldots, m+1$. Moreover, let us define $(\vec{u}, \vec{v})_S = \int_S \vec{u} \cdot \vec{v} \, dS$ with $S \subset \Omega$ and $(f,g)(0,L) = \int_0^L f(x)g(x) \, dx$. We have:

**Proposition 7.** Let $(k_h, \vec{v}_h) \in \mathbb{R} \times Y_{h,S_i}$ be an eigenpair of Problem 2 formulated in the domain $S_i^m, i = 1, \ldots, m$, with $Y_{h,S_i}$ as FE space and let $\vec{u}_h = \vec{v}_h$ in $S_i^m$ and $\vec{u}_h = 0$ in $\Omega \setminus S_i^m$. Then $(k_h, \vec{v}_h) \in \mathbb{R} \times Y_h$ is an eigenpair of Problem 2 with $Y_h$ as FE space.

**Proof.** $\vec{u}_h$ is tangentially continuous. Hence it belongs to $Y_h$. Moreover, $(\nabla \times \vec{u}_h, \nabla \times \vec{w}_h) = (\nabla \times \vec{v}_h, \nabla \times \vec{w}_h|_{S_i^m}) = k_h^2 (\vec{v}_h, \vec{w}_h|_{S_i^m}) = k_h^2 (\vec{u}_h, \vec{w}_h) \forall \vec{w}_h \in Y_h$.

Propositions 7 and 6 means that some eigenpairs of Problem 2 can be deduced by studying the 1D eigenproblem formulated on $S_i^m$ with $Y_{h,S_i}$ as FE space.

From the analytical expressions of $\vec{w}_2$ and $\vec{w}_4$ we can see that any vector field $\vec{v}_h \in Y_{h,S_i}$ can be written as $\vec{v}_h = \vec{e}_x \, g(y) \, r_i^m(x)$, where $g(y) \in L^2_D$ and $r_i^m(x) = (2x - x_i^m - x_{i+1}^m)/(x_{i+1}^m - x_i^m)$.

We deduce:

**Proposition 8.** Any eigenpair $(k_h, \vec{v}_h) \in \mathbb{R} \times Y_{h,S_i}$ of Problem 2 formulated in $S_i^m, i = 1, \ldots, m$, with $Y_{h,S_i}$ as FE space is such that $\vec{v}_h = \vec{e}_x \, g(y) \, r_i^m(x)$, with $k_h$ and $g(y) \in L^2_D$ determined by the following 1D eigenproblem:

**Problem 3.** Find $k_h \in \mathbb{R}$ and $g \in L^2_D$, $g \neq 0$, such that $(\frac{d}{dy} \frac{dg}{dx})(0,H) - k_h^2 (g,f)(0,H) = 0 \forall f \in L^2_D$.

**Proof.** For any $\vec{v}_h \in Y_{h,S_i}$ we know that $\vec{v}_h = \vec{e}_x \, f(y) \, r_i^m(x), f \in L^2_D$. Thus we have $(\nabla \times \vec{v}_h, \nabla \times \vec{w}_h) = (r_i^m(x), r_i^m(x))(\frac{dx}{dy}, \frac{df}{dy})(0,H) + (\vec{v}_h, \vec{w}_h) = (r_i^m(x), r_i^m(x))(g, f)(0,H)$. Thus, some eigenpairs of Problem 2 can be found by studying a one-dimensional scalar eigenproblem.

The solution to Problem 3 is analytically provided in [20]. The $j$-th eigenpair $(k_{bj}^2, g_j(y))$ is such that $g_j(y) = \sin(j \pi (y - 1)/m)$, $m \geq 2$, $j = 1, \ldots, m - 1$, $p = 1, \ldots, m + 1$ and $k_{bj}^2 = 6(m/H)^2(1 - \cos(j \pi /m))/(2 + \cos(j \pi /m))^{-1}$.
If we now consider the vector field $\vec{v}_m^j$ defined by
$$\vec{v}_m^j = \vec{e}_x (-1)^{i+1} g_j(y) r_0^m(x), \quad i = 1, \ldots, m,$$
$$j = 1, \ldots, m-1, \quad m \geq 2,$$
by using Propositions 6, 7 and 8 we obtain that $(k_{j}^2, \vec{v}_m^j) \in \mathbb{R} \times V_h$ is an eigenspace of Problem 2. Moreover we have:

**Proposition 9.** The elements of the sequence $\{\vec{v}_m^j\}_{i \in \mathbb{N}}$ are such that $(\vec{v}_m^i, \vec{v}_m^l) = 0$ whenever $i \neq l$.

**Proof.** Without any loss of generality, let us suppose $i > l$. Thus any vertical strip $S_{l}^p$ of the coarser mesh is the union of an even number of strips of the finer mesh. From the analytical expression of the vector fields making up the sequence we may note that $\vec{v}_m^j \bigg|_{S_{l}^p}$ is the product of a function of the $y$ coordinate by an odd function of the $x$ coordinate (odd with respect to the center of $S_{l}^p$) while $\vec{v}_m^i \bigg|_{S_{l}^p}$ is the product of a function of the $y$ coordinate by an even function of the $x$ coordinate (even with respect to the center of $S_{l}^p$). Thus the integral involved in the scalar product definition is always zero.

Proposition 9 exhibits an infinite set of nonconverging sequences $\{\vec{v}_m^i\}_{i \in \mathbb{N}}$ that violate (3) and consist of mutually orthogonal eigenfields. It is easy to verify that the corresponding sequences of eigenvalues do converge to physically meaningful eigenvalues (those of the modes $TE_{j0}$ or $TE_{0j}$). On the other hand, as already pointed out, the corresponding sequences of eigenfields do not converge at all. Thus all these modes are spurious. In particular, condition (NPS) of [12] is not satisfied. It can be shown, however, that the spectrum is correct and no eigenfield is missed (i.e. (NPS), (CS), (CE) and (IDK) of [12] are all satisfied).

To the best of authors’ knowledge this is the first time that such a subtle behaviour in which all the eigenfrequencies are correct but some eigenfields are spurious has been found. The possibility of such a behaviour was pointed out in [12], but at that time we did not know any example of it. Having found this example stresses the importance of (NPE) in Definition 4.7 of [12]. Numerical simulation of the present example has confirmed the predicted behaviour.

**Remark 3.** It is easy to verify that, on the contrary, when the $Q_{11}$ edge elements are used to find the transverse magnetic field of TM modes at cutoff we find a more usual situation in which the approximation violates condition (NPS) since all eigenvalues corresponding to the nonphysical modes $TM_{p0}$ and $TM_{0p}$ are calculated.

**Remark 4.** In order to fit the limited space allowed we have exploited the same sequence to show both that (3) is violated and spurious modes occur. Sequences that violate (3), but do not consist of eigenfields, do exist, however.

**Remark 5.** By Proposition 5, both (4) and (2) are satisfied in this counterexample. Hence, it disproves both versions of Proposition 1.

### 5 Conclusions

The need to replace the widely accepted theory on spurious modes with the new one proposed by the present authors is stressed. This is done by providing two counterexamples to the old theory which do not conflict with the new one. One of them shows a new and more subtle behaviour of spurious modes.

### References

“Counterexamples to the currently accepted explanation for spurious modes...”


