An ill posed waveguide discontinuity problem involving metamaterials with impedance boundary conditions on the two ports

Mirco Raffetto

October 25, 2006

Abstract

In this paper it is shown, for the first time to the best of author’s knowledge, that the presence of double negative metamaterials can affect the traditional well posedness of time-harmonic electromagnetic boundary value problems, even when absorbing boundary conditions are considered on part of the boundary. This is done by considering a simple waveguide discontinuity problem with traditional impedance boundary conditions on the two ports. The theoretical results are then used to introduce some doubts on the reliability of models with sharp interfaces between lossless double positive and double negative materials, when the latter are constructed with arrangements of normal materials, and to explain the anomalous behaviour of many numerical simulators when double negative metamaterials are involved.

Keywords - Waveguide, waveguide discontinuity, electromagnetic boundary value problems, metamaterials, electromagnetic field theory, computational electromagnetics, finite element method.

1 Introduction

According to Hadamard [1], mathematical models of physical phenomena should be well posed, that is they should have the properties that a solution exists, is unique and depends continuously on the data. The importance of part of these considerations is well known to all members of the electromagnetic research community. As a matter of fact, many fundamental textbooks [2], [3], [4], [5], address the issue of the uniqueness of the solution of time-harmonic electromagnetic boundary value problems with its fundamental byproducts, for example in terms of equivalence theorems. Some more recent contributions extend the traditional results in order to cover cases of more practical interest in applications, such as those with dielectric discontinuities [6], [7], [8], [9]. However, in the last two decades also the other aspects of Hadamard’s definition have become more and more important. For example microwave engineers are now used to designing components or devices by exploiting the power of numerical simulators but these deal just with mathematical models retaining only a part of the set of physical principles which could guarantee, for example, the existence of the solution. Finally, many researchers involved in the study of electromagnetic inverse problems are well aware of the importance of Hadamard’s definition even though they are used to dealing with problems that do not satisfy the definition and that, for this reason, are said to be ill posed [10], [11], [8].

It has just been pointed out that electromagnetic inverse problems are often ill posed. On the contrary, direct problems are or can be made well posed with the most appropriate and physically sound choice of the boundary conditions. When only traditional double positive media [12] are involved the only well known ill posed time-harmonic electromagnetic boundary value problems

*Department of Biophysical and Electronic Engineering, University of Genoa, Via Opera Pia 11a, I–16145, Genoa, Italy, email: raffetto@dibe.unige.it
are defined by enforcing source terms on ideal cavity resonators having a mode resonating at
the working frequency of the time-harmonic sources themselves. In order to avoid this kind of ill
posedness, when all media are double positive, the requirements are well established. In particular,
it is sufficient to consider at least losses in a part (of nonzero measure) of the dielectric involved or
a part (of nonzero measure) of the boundary should be able to absorb power [8]. Unfortunately,
when metamaterials [12] are involved the situation is more intricate. The first results on the well
posedness of direct problems involving metamaterials are provided in [13], [14] and require the
presence of losses on the whole boundary and in the dielectrics filling entire subregions. Even
the less restrictive of the two results [14] is too restrictive for many applications and, for their
impact on the reliability of numerical simulators already pointed out, a generalization of these
conclusions is of great interest. Unfortunately we have still to understand why satisfactory results
in the presence of metamaterials are lacking. Is this a consequence of the technical difficulty
in generalizing the proofs available when only double positive media are involved or is there a
difference in the results which can be achieved?

This paper shows that the presence of double negative metamaterials [12] heavily affects the
results of well posedness which can be obtained. In particular it is shown, for the first time to
the best of author’s knowledge, that by using the typical formulation adopted for the study of
waveguide discontinuity problems, with absorbing boundary conditions on the two ports which
results in a well posed problem when just double positive media are involved [15], [8], an ill posed
problem is obtained when some (infinite, actually) particular configurations of lossless double
positive media and double negative metamaterials are considered, independently of the value of
the working frequency. The technical reason for such an anomalous result is shown to be due to
the fact that for particular combinations of the dielectrics at any fixed frequency infinite surface
waves can exist.

The existence of time-harmonic electromagnetic driven problems involving double negative
metamaterials for which the most reliable mathematical formulation is ill posed have a significant
practical consequence. As a matter of fact, most discretization methods applied to such driven
problems result in algebraic problems that are ill conditionned. Thus the approximations provided
by numerical simulators are expected to be unreliable. Another reason of unreliability of the
results of numerical simulators, which can appear even for well posed problems, is deduced from
the behaviour of the surface waves defined in our theoretical development. The two effects are
shown to have a terrible impact on the possibility of obtaining reliable (time-harmonic) numerical
simulators with no restrictions on the dielectric parameters. Finally, it is pointed out that our
analysis could provide a justification for the numerical effects described in [16], [17], [18] and [19].

Our study is carried out by assuming that double negative metamaterials exist without any
consideration on the technology which allows their construction, according to the so-called “what-
if” approach [20]. Many interesting studies have been based on such an approach [21], [22].
However, our results show that the models presenting sharp interfaces between lossless double
positive and double negative materials could be unreliable, as far as double negative metamaterials
are obtained by using inclusions of double positive media in double positive host materials.

This paper is organized as follows. In Section 2 the class of simple time-harmonic electromagnetic
boundary value problems considered in this work is introduced. The ill posedness of some
problems of this class is discussed in Section 3. Finally, in Section 4 a discussion on the practical
consequences of our theoretical developments is provided.

2 Problem definition and mathematical formulation

The geometry and the dielectric configuration of the simple problems considered in this paper is
shown in Figure 1. A rectangular waveguide is filled with two homogeneous and lossless media.
No electric or magnetic sources are present in the waveguide.

By using the cartesian reference system shown in Figure 1, the medium occupying region
\( \Omega_1 = \{(x, y, z) \in \mathbb{R}^3 : x \in (0, a), y \in (0, b), z \in (z_1, 0)\} \) is assumed to be a double positive dielectric
[12] characterized by an effective dielectric permittivity and an effective magnetic permeability
An ill posed waveguide discontinuity problem involving metamaterials with... 

\[ \varepsilon_1 > 0 \text{ and } \mu_1 > 0, \text{ respectively. Analogously, } \varepsilon_2 \in \mathbb{R} \text{ and } \mu_2 \in \mathbb{R} \text{ denote the effective dielectric permittivity and the effective magnetic permeability of the medium in region } \Omega_2 = \{(x, y, z) \in \mathbb{R}^3 : x \in (0, a), y \in (0, b), z \in (0, z_2)\}. \]

However, the medium in region 2 is allowed to be made up of either a double positive dielectric or a double negative metamaterial [12]. The interface at the dielectric discontinuity is denoted by \( \Gamma_{\text{int}} \).

As far as the boundary conditions are concerned, the walls of the waveguide will be assumed to be made up of a perfect electric conductor and will be denoted by \( \Gamma_{\text{pec}} \). On the contrary, on the two ports at \( z = z_2 \) and \( z = z_1 \), denoted respectively by \( \Gamma_2 \) and \( \Gamma_1 \), possibly inhomogeneous impedance boundary conditions will be enforced [15], [13], [23]. In particular, we will assume that the incident field is the dominant mode above cutoff impinging on the port at \( z = z_1 \). Thus, the impedance boundary condition will be inhomogeneous only at \( z = z_1 \).

According to the previous hypothesis the angular working frequency \( \omega \) is assumed to satisfy

\[ \omega > \frac{\pi}{a \sqrt{\varepsilon_1 \mu_1}}. \]

As a consequence of our assumptions and by taking account of the fact that the unit vector \( \mathbf{n} \) normal to the boundary surfaces is equal to \( \hat{z} \) on \( \Gamma_2 \) and \( \Gamma_{\text{int}} \) and to its opposite on \( \Gamma_1 \), the mathematical formulation of the simple problems of interest can be the following:

\[
\begin{align*}
\nabla \times \mathbf{H} - j\omega \varepsilon_1 \mathbf{E} &= 0 & \text{in } \Omega_1 \\
\nabla \times \mathbf{E} + j\omega \mu_1 \mathbf{H} &= 0 & \text{in } \Omega_1 \\
\nabla \times \mathbf{H} - j\omega \varepsilon_2 \mathbf{E} &= 0 & \text{in } \Omega_2 \\
\nabla \times \mathbf{E} + j\omega \mu_2 \mathbf{H} &= 0 & \text{in } \Omega_2 \\
\mathbf{E} \times \mathbf{n} &= 0 & \text{on } \Gamma_{\text{pec}} \\
\mathbf{H}|_{z=0-} \times \hat{z} &= \mathbf{H}|_{z=0+} \times \hat{z} & \text{on } \Gamma_{\text{int}} \\
\mathbf{E}|_{z=0-} \times \hat{z} &= \mathbf{E}|_{z=0+} \times \hat{z} & \text{on } \Gamma_{\text{int}} \\
\mathbf{H} \times \hat{z} + Y_{c1} (\hat{z} \times \mathbf{E} \times \hat{z}) &= \mathbf{f}_{R1} & \text{on } \Gamma_1 \\
\mathbf{H} \times \hat{z} - Y_{c2} (\hat{z} \times \mathbf{E} \times \hat{z}) &= 0 & \text{on } \Gamma_2
\end{align*}
\]

(1)

where \( Y_{c1} = \frac{1}{Z_{c1}} = \frac{\alpha}{\omega \mu_1} \in \mathbb{R}, \ i = 1, 2, \) is the dominant mode admittance [24] and

\[
f_{R1} = 2 \sqrt{\frac{2}{ab}} Y_{c1} \sin \left( \frac{\pi x}{a} \right) e^{-j \beta_1 z_1} \hat{y} V_i^+ \]

(2)
is the usual [15] (pp. 357-361) known term enforced for the calculation of the field in the presence of an incident wave given by the dominant mode ($V_i^* \in \mathbb{C}$ is the amplitude coefficient) impinging on the port at $z = z_1$. In the above two expressions, the phase constants $\beta_i, \ i = 1, 2$, of the dominant mode are real and are given by [24]

$$\beta_i = \alpha_i \sqrt{\omega^2 \varepsilon_i \mu_i - \frac{\omega^2}{a^2}} \quad \text{(3)}$$

where $\alpha_i = 1$ if region $i$ is occupied by a double positive medium and $\alpha_i = -1$ if region $i$ is occupied by a double negative metamaterial [25]. It is important to note that the sign of $Y_{ci}$ is positive for double positive and double negative media [25], so that, in any case, the impedance boundary condition is actually an absorbing boundary condition [15].

3 Considerations on the ill posedness of some of the problems of interest

The results on the well posedness of boundary value problems are particularly useful and can be exploited by other researchers when they are provided for an entire and sufficiently general class of problems. On the contrary, a result on the ill posedness could be useful even if it is deduced for a set of particular examples, especially if no comparable results are known to the scientific community. Moreover, in order to prove the ill posedness it is sufficient to show that one of the three aspects considered in the definition (i.e., existence, uniqueness and continuous dependence) does not hold true. On the basis of these considerations in the following the existence of the solution is just shown for a particular inhomogeneous term in the boundary conditions, the uniqueness of the solution is discussed in more generality but in a concise way and we focus our attention on the continuous dependence of the solution on the data. In particular, we prove by direct calculations that for some of the models of interest such continuous dependence does not hold true.

3.1 Existence of a solution for the problems of interest

The problem defined in Section 2 with $f_{R1}$ defined by (2) is trivially solved by the appropriate combination of forward and backward traveling waves of the fundamental mode in the considered rectangular waveguide. In particular it is easy to verify that the fields defined as $E = E_y \hat{y}$ and $H = H_x \hat{x} + H_z \hat{z}$, where

$$E_y = -\sqrt{\frac{2}{ab}} V_1^+ \sin \left(\frac{\pi x}{a}\right) f_1(z) \quad \text{(4)}$$

$$H_x = \sqrt{\frac{2}{ab}} V_1^+ \sin \left(\frac{\pi x}{a}\right) f_2(z) \quad \text{(5)}$$

$$H_z = \frac{\pi}{a} \sqrt{\frac{2}{ab}} V_1^+ \cos \left(\frac{\pi x}{a}\right) f_3(z) \quad \text{(6)}$$

with

$$f_1(z) = \begin{cases} e^{-j\beta_1 z} + \frac{Z_{12}}{Z_{21} + Z_{c1}} e^{j\beta_1 z} & \text{in medium 1} \\ \frac{Z_{21}}{Z_{12} + Z_{c1}} e^{-j\beta_2 z} & \text{in medium 2} \end{cases} \quad \text{(7)}$$

$$f_2(z) = \begin{cases} \frac{1}{z + 1} e^{-j\beta_1 z} - \frac{Z_{21}}{Z_{12} + Z_{c1}} e^{j\beta_1 z} & \text{in medium 1} \\ \frac{1}{z + 2} Z_{12} e^{-j\beta_2 z} & \text{in medium 2} \end{cases} \quad \text{(8)}$$

$$f_3(z) = \begin{cases} \frac{1}{z + \mu_1} e^{-j\beta_1 z} + \frac{Z_{21}}{Z_{12} + Z_{c1}} e^{j\beta_1 z} & \text{in medium 1} \\ \frac{1}{z + \mu_2} Z_{21} e^{-j\beta_2 z} & \text{in medium 2} \end{cases} \quad \text{(9)}$$

satisfy problem (1) with forcing term defined by (2). Thus the existence of a solution for the problem of interest with $f_{R1}$ defined by (2) is trivially guaranteed.
3.2 Uniqueness of the solution

Problem (1) is linear and the uniqueness of the solution can be faced by showing that the corresponding homogeneous (i.e. $f_R = 0$) problem admits only the trivial solution. This is classically proved by using Poynting’s theorem and the unique continuation principle [2], [26], [7]. In particular, by using Poynting’s theorem together with the fact that $f_R = 0$, no volumetric sources are present and the dielectric media are lossless one easily deduces that ($^*$ denotes the complex conjugate)

$$0 = \int_{\Gamma_1 \cup \Gamma_2} \text{Re} ((\mathbf{E} \times \mathbf{H}^*) \cdot \mathbf{n}) \, dS = \int_{\Gamma_1 \cup \Gamma_2} \text{Re} ((\mathbf{H}^* \times \mathbf{n}) \cdot \mathbf{E}) \, dS = Y_{c1} \int_{\Gamma_1} (|\mathbf{n} \times \mathbf{E} \times \mathbf{n}|^2) \, dS + Y_{c2} \int_{\Gamma_2} (|\mathbf{n} \times \mathbf{E} \times \mathbf{n}|^2) \, dS. \quad (10)$$

According to the final remark of Section 2, $Y_{c1}$ and $Y_{c2}$ are real and have always the same sign. Thus one obtains $\mathbf{n} \times \mathbf{E} \times \mathbf{n} = 0$ on $\Gamma_i$, $i = 1, 2$. By using the two homogeneous impedance boundary conditions one then deduces $\mathbf{H} \times \mathbf{n} = 0$ on $\Gamma_i$, $i = 1, 2$. Now the considerations reported on p. 135 of [2] above the statement of Theorem 34 allow us to conclude that $\mathbf{E}|_{\Omega_i}$ and $\mathbf{H}|_{\Omega_i}$ vanish identically, $i = 1, 2$.

It is important to point out that the same result can be obtained in a more general context by using the considerations reported in [8] (pp. 92-95). In particular this more general procedure deals with more general spaces of vector fields (i.e. $H(\text{curl}, \Omega)$ or its subspaces) and can be applied even to the variational formulation of the problem of interest, which, for example, provides the starting point for the development of numerical simulators based on the finite element method [27], [13], [28].

3.3 Continuous dependence of the solution on the data

In general a solution is said to depend continuously on data if “small” (in an appropriate norm) changes in the data result in “small” (in an appropriate norm) changes in the solution. For linear problems a solution is equivalently said to depend continuously on data if “small” (in an appropriate norm) data result in “small” (in an appropriate norm) solutions.

For Problem (1) the data are the forcing terms in the impedance boundary conditions on the two ports, i.e., $f_{R1}$ on $\Gamma_1$ and 0 on $\Gamma_2$, and the solutions are given by the electric and magnetic fields. The most simple norms which can be used to evaluate the amplitude of these quantities are the $L^2$ norms on $\Gamma_1 \cup \Gamma_2$ and on $\Omega_1 \cup \Omega_2$.

Thus, for the problem of interest, the solution depends continuously on the data if for any inhomogeneous boundary term given by $f_{R1}$ on $\Gamma_1$ and $f_{R2}$ on $\Gamma_2$ there exists a solution $(\mathbf{E}, \mathbf{H})$ of problem (1) and a constant $C > 0$ independent of $f_{R1}$, $f_{R2}$, $\mathbf{E}$ and $\mathbf{H}$ such that,

$$\int_{\Omega_1 \cup \Omega_2} \mathbf{E} \cdot \mathbf{E}^* \, dV + \int_{\Omega_1 \cup \Omega_2} \mathbf{H} \cdot \mathbf{H}^* \, dV \leq C \left( \int_{\Gamma_1} f_{R1} \cdot f_{R1}^* \, dS + \int_{\Gamma_2} f_{R2} \cdot f_{R2}^* \, dS \right). \quad (11)$$

Such an inequality is guaranteed by the results in [8], [29] and [30] when only double positive media are involved so that the problem of interest is well posed under this additional hypothesis.

In order to show that (11) may not hold true anymore when double negative metamaterials
are involved together with double positive media let us define the fields

\[ E_{mn} = \begin{cases} \frac{1}{k_{cm_n}} \left( \frac{m\pi}{a} \right) \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) f_4(z) \\ + \frac{1}{k_{cm_n}} \left( \frac{n\pi}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) f_4(z) \\ + 2 \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) f_5(z) \end{cases} \]  

(12)

and

\[ H_{mn} = \begin{cases} j\omega \frac{k_{cm_n}}{k_{cm_n}} \left( \frac{n\pi}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) f_6(z) \\ -j\omega \frac{k_{cm_n}}{k_{cm_n}} \left( \frac{n\pi}{a} \right) \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) f_6(z) \end{cases} \]  

(13)

where \( m, n \in \mathbb{N}, m > 0, n > 0, k_{cm_n}^2 = \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \).

\[ f_4(z) = \begin{cases} V_{1}^- e^{\gamma_{1mn} z} & \text{in medium 1} \\ -V_{2}^- e^{-\gamma_{2mn} z} & \text{in medium 2} \end{cases} \]  

(14)

\[ f_5(z) = \begin{cases} V_{1}^+ e^{\gamma_{1mn} z} & \text{in medium 1} \\ V_{2}^+ e^{-\gamma_{2mn} z} & \text{in medium 2} \end{cases} \]  

(15)

\[ f_6(z) = \begin{cases} \varepsilon_1 V_{1}^- e^{\gamma_{1mn} z} & \text{in medium 1} \\ \varepsilon_2 V_{2}^- e^{-\gamma_{2mn} z} & \text{in medium 2} \end{cases} \]  

(16)

with \( \gamma_{1mn} \) and \( \gamma_{2mn} \) representing the propagation constants of TM_{mn} or TE_{mn} modes in rectangular waveguides [24].

The above fields are the usual TM_{mn} modes in rectangular waveguides [24]. It is well known [24] that Maxwell’s equations in the two regions are satisfied together with the boundary conditions on \( \Gamma_{pec} \) for any \( m, n \in \mathbb{N}, m > 0, n > 0 \).

From (12), (13), (14) and (16) we deduce that, in order to guarantee the continuity of the tangential components of \( E_{mn} \) and \( H_{mn} \) on \( \Gamma_{int} \), it is necessary and sufficient to enforce

\[ \begin{cases} -V_{2mn}^+ \gamma_{2mn} = V_{1mn}^- \gamma_{1mn} \\ \varepsilon_2 V_{2mn}^- = \varepsilon_1 V_{1mn}^- \end{cases} \]  

(17)

where \( \gamma_{1mn} \) and \( \gamma_{2mn} \) are not independent according to the previous definitions.

For reasons that will become evident later on, we are particularly interested only on the evanescent modes which decay exponentially as we move away from the interface. In particular, for any working angular frequency \( \omega \) we consider the modes with \( m \) or \( n \) sufficiently large to give \( k_{cm_n}^2 > \omega^2 \varepsilon_i \mu_i, i = 1, 2 \), so that \( \gamma_{1mn}^2 = k_{cm_n}^2 - \omega^2 \varepsilon_i \mu_i > 0, i = 1, 2 \).

It is interesting to note that (17) does not admit any “surface wave” solution if \( \varepsilon_1 \) and \( \varepsilon_2 \) have the same sign, since a “surface wave” requires [31] \( \gamma_{1mn}^2 > 0, i = 1, 2 \). On the contrary, when \( \varepsilon_1 \) and \( \varepsilon_2 \) have opposite signs a “surface wave” can be found provided that \( m, n, \omega, \varepsilon_1, \varepsilon_2, \mu_1 \) and \( \mu_2 \) are such that

\[ -\varepsilon_2 \gamma_{1mn} = \varepsilon_1 \gamma_{2mn} \]  

(18)

is satisfied.

From (18) and the definitions of \( \gamma_{imn}, i = 1, 2 \), we deduce that, under the condition that the sign of \( \varepsilon_1 \) is opposite to the sign of \( \varepsilon_2 \), a “surface wave” is possible when \( m, n, \omega, \varepsilon_1, \varepsilon_2, \mu_1 \) and \( \mu_2 \) are such that

\[ \left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2 \left( \varepsilon_2^2 - \varepsilon_1^2 \right) = \omega^2 \varepsilon_1 \varepsilon_2 (\mu_1 \varepsilon_2 - \mu_2 \varepsilon_1). \]  

(19)
Different possibilities could be of interest. For example, if $\mu_1 \varepsilon_2 - \mu_2 \varepsilon_1 = 0$ and $\varepsilon_2^2 - \varepsilon_1^2 \neq 0$ or if $\mu_1 \varepsilon_2 - \mu_2 \varepsilon_1 \neq 0$ and $\varepsilon_2^2 - \varepsilon_1^2 = 0$ no TM$_{mn}$ "surface wave" satisfies Maxwell’s equations in the two regions together with the boundary conditions on $\Gamma_{pec}$ and $\Gamma_{int}$. On the contrary, when $\mu_1 \varepsilon_2 - \mu_2 \varepsilon_1 \neq 0$ and $\varepsilon_2^2 - \varepsilon_1^2 \neq 0$ a TM$_{mn}$ "surface wave" could satisfy Maxwell’s equations in the two regions together with the boundary conditions on $\Gamma_{pec}$ and $\Gamma_{int}$ only at a particular value of the angular frequency. However, the most interesting case is when $\mu_1 \varepsilon_2 - \mu_2 \varepsilon_1 = 0$ and $\varepsilon_2^2 - \varepsilon_1^2 = 0$. In this case (remember that $\varepsilon_2$ and $\varepsilon_1$ must have opposite signs) $\varepsilon_2 = -\varepsilon_1$, $\mu_2 = -\mu_1$ (the media are conjugate or complementary, according to [32]) and for any given angular frequency infinite value of the working angular frequency $\omega$ is found as an infinite set of "surface waves" on $\Gamma_{mn}$.

The existence of such an infinite set of "surface waves" is, as we are going to show, the ultimate reason for the ill posedness of the corresponding boundary value problem, independently of the value of the working angular frequency $\omega$.

The boundary values of the TM$_{mn}$ "surface waves" on $\Gamma_i$, $i = 1, 2$, are fundamental for our theoretical developments. With this aim, let us calculate the right-hand sides of the last two boundary conditions of problem (1) when the generic fields $E$ and $H$ are substituted by $E_{mn}$ and $H_{mn}$, respectively:

$$f_{R1mn} = H_{mn} |_{z = z_1} \times \hat{z} + Y_{c1} (\hat{z} \times E_{mn} |_{z = z_1} \times \hat{z}) = (\hat{x} \left( \frac{m\pi}{a} \right) \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) + \hat{y} \left( \frac{n\pi}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) + \frac{1}{k_{c_{mn}}^+} (j\omega \varepsilon_1 - Y_{c1} \gamma_{1mn})$$

and

$$f_{R2mn} = H_{mn} |_{z = z_2} \times \hat{z} - Y_{c2} (\hat{z} \times E_{mn} |_{z = z_2} \times \hat{z}) = (\hat{x} \left( \frac{m\pi}{a} \right) \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) + \hat{y} \left( \frac{n\pi}{b} \right) \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) + \frac{1}{k_{c_{mn}}^-} (Y_{c2} \gamma_{2mn} - j\omega \varepsilon_2)$$

In all the previous considerations it is obvious that for any given $m$ and $n$ (sufficiently large), the surface waves fields $E_{mn}$ and $H_{mn}$ give the unique solution to problem (1) when the data are given by $f_{R1mn}$ and $f_{R2mn}$.

After some calculations we deduce that

$$\int_{\Gamma_1} f_{R1mn} \cdot f_{R1mn}^* dS + \int_{\Gamma_2} f_{R2mn} \cdot f_{R2mn}^* dS = \frac{ab}{4k_{c_{mn}}^2} (|V_{1mn}^-|^2 e^{2\gamma_{1mn} z_1 |j\omega \varepsilon_1 - Y_{c1} \gamma_{1mn}|^2} + |V_{2mn}^-|^2 e^{-2\gamma_{2mn} z_2 |Y_{c2} \gamma_{2mn} - j\omega \varepsilon_2|^2})$$

Let us remember that for conjugate media $Y_{c1} = Y_{c2}$, $\varepsilon_1 = -\varepsilon_1$, $\gamma_{1mn} = \gamma_{2mn}$ and $V_{2mn}^- = -V_{1mn}^-$. If we choose $n = 1$ and $m$ arbitrarily large we deduce, for any fixed $\omega$, $\gamma_{1mn} = \gamma_{2mn} \approx \frac{1}{2\mu_0}$, $|j\omega \varepsilon_1 - Y_{c1} \gamma_{1mn}|^2 \approx Y_{c1}^2 \left( \frac{m\pi}{a} \right)^2$, $|Y_{c2} \gamma_{2mn} - j\omega \varepsilon_2|^2 \approx Y_{c2}^2 \left( \frac{m\pi}{a} \right)^2 = Y_{c1}^2 \left( \frac{m\pi}{a} \right)^2$ and $k_{c_{mn}}^2 \approx \left( \frac{m\pi}{a} \right)^2$. 
Thus, for $n = 1$ and $m$ arbitrarily large

$$
\int_{\Gamma_1} f_{R1m1} \cdot f_{R1m1}^* dS + \int_{\Gamma_2} f_{R2m1} \cdot f_{R2m1}^* dS \\
\simeq \frac{ab}{4} |V_{m1}^-|^2 |V_{c1}|^2 \left( e^{2\pi m z_1} + e^{-2\pi m z_2} \right).
$$

Let us now consider the integral of the solution on $\Omega_1 \cup \Omega_2$. We have

$$
\int_{\Omega_1 \cup \Omega_2} E_{mn} \cdot E^*_{mn} dV \\
+ \int_{\Omega_1 \cup \Omega_2} H_{mn} \cdot H^*_{mn} dV \geq \int_{\Omega_1 \cup \Omega_2} E_{zm1} \cdot E^*_{zm1} dV.
$$

But from (12) and (15) we deduce that

$$
\int_{\Omega_1 \cup \Omega_2} E_{zm1} \cdot E^*_{zm1} dV = \\
\frac{ab}{8} \left( \frac{|V_{m1}^-|^2}{\gamma_{mn1}} (1 - e^{2\gamma_{2m1} z_1}) + \frac{|V_{m1}^+|^2}{\gamma_{2m1}} (1 - e^{-2\gamma_{2m1} z_2}) \right).
$$

Again, by choosing $n = 1$ and $m$ arbitrarily large we deduce, for conjugate media and for any fixed $\omega$,

$$
\int_{\Omega_1 \cup \Omega_2} E_{zm1} \cdot E^*_{zm1} dV \simeq \frac{ab}{4} \frac{a}{m\pi} |V_{m1}^-|^2.
$$

The right-hand side of (23) goes to zero as $m \to \infty$ much more rapidly then (26). Thus, by taking account of (24), for any $C > 0$ we can find a value of $m$ sufficiently large such that a condition like (11) cannot hold true for the forcing term $(f_{R1m1}, f_{R2m1})$ and the corresponding solution $(E_{m1}, H_{m1})$. By definition this means that the solution of problem (1) does not depend continuously on the data whenever $\varepsilon_2 = -\varepsilon_1$ and $\mu_2 = -\mu_1$. If these conditions are satisfied our conclusion holds true independently of the value of the working frequency. Finally, it is easy to verify that the same conclusion holds true even if a stronger norm is used for the data, such as that of $(H^1(\Gamma_i))^2$.

### 4 Theoretical considerations and practical implications

As already pointed out it is well known that a time-harmonic electromagnetic driven problem can result to be ill posed when the corresponding homogeneous problem defines a cavity having a mode resonating at the working frequency of the driven problem of interest. This can happen when no losses are present in the dielectric media involved and the boundary conditions are enforced in terms of the tangential components of the electric field or in terms of the tangential components of the magnetic field or even in terms of the tangential components of the electric field on part of the boundary and in terms of the tangential components of the magnetic field on the rest of the boundary [7].

The novelty of the previous deduction is that there exist time-harmonic electromagnetic driven problems that result in ill posed mathematical models even when absorbing boundary conditions are enforced. Up to now it was believed, to the best of author’s knowledge, that such formulations were able to guarantee the well posedness. Actually, for problems involving just double positive media this belief is supported by mathematical proofs [8] as those sketched in Sections 3.1, 3.2 and at the beginning of Section 3.3. Unfortunately, our result shows that such a consideration cannot be extended to generic problems involving double positive and double negative metamaterials.
A byproduct of our analysis is that by using just double positive materials in a rectangular waveguide it is not possible to obtain a dielectric configuration like that studied in the previous section with conjugate media since the electromagnetic problem should be, at the same time, well posed and ill posed. It should be well posed when the fine structure of the double positive inclusions are considered and it should be ill posed if these inclusions were able to generate a double negative metamaterial with geometric and dielectric features like that considered above. It should be noted that the previous consideration provides just a theoretical limitation since we have deduced that lossless double positive inclusions in a lossless double positive host medium cannot be used to obtain, in a rectangular waveguide and at any frequency, a homogeneous and lossless double negative metamaterial with a step discontinuity at the double positive-double negative interface.

However, for microwave engineers the most important practical effect of our result is that we have lost the possibility of obtaining a priori reliability results for many numerical simulators of time-harmonic electromagnetic driven problems if no restrictions on the problems to which the result of reliability applies are considered. In particular, the available results for problems involving just double positive media cannot be extended to cover cases involving double negative metamaterials without limitations on the dielectric characteristics of the media themselves. This is due to the fact that many accurate discretization techniques applied to an ill posed linear problem, like problem (1) in the presence of conjugate dielectrics, require the solution of a linear algebraic system of equations characterized by an ill conditioned matrix, independently of round-off errors and, if sufficiently large, of the number of degrees of freedom.

For example, for numerical simulators based on the (time domain versions are excluded from our considerations) finite element method [15], which, in turn, are based on variational formulations [13] equivalent to formulations like problem (1), the general, physically sound, satisfactory and reassuring results of reliability stated in [8] cannot be extended with no additional hypotheses to cover cases involving double negative metamaterials. The results in [8] essentially say that “losses” in part of the dielectrics or on part of the boundary are sufficient to give the well posedness of continuous problems and the convergence of their finite element approximations. When double negative metamaterials are involved together with double positive media some additional “ad hoc” restrictions will be necessarily introduced in order to obtain the same result. In this sense the results obtained in [13] and [14] are a couple of significant examples.

One could be tempted to say that the case with conjugate dielectrics is known in advance to be pathological and could be easily identified. This would be the right approach if we restricted our interest just to the continuous problem. However the most important practical effect of our analysis is on numerical simulators and for them, unfortunately, instability can occur also in many other cases. As a matter of fact, if for a particular dielectric configuration and at a particular working frequency a \( \text{TM}_{mn} \) “surface wave” satisfies Maxwell’s equations together with the boundary conditions on \( \Gamma_{pec} \) and \( \Gamma_{int} \) then its exponential attenuation as we move away from the interface makes this electromagnetic field so small at the two ports that it “almost” satisfies homogeneous impedance boundary conditions. This is not a problem for the continuous formulation since from an analytical point of view the homogeneous impedance boundary conditions are not “exactly” satisfied, independently of the distance of the two ports from the interface, but it is a problem for many numerical methods since the superposition of this kind of non trivial field to any solution does not significantly affect the data on \( \Gamma_1 \) and \( \Gamma_2 \) of the problem and, if the two ports are at a sufficient distance, its effect on these data could be completely neglected by round-off errors. From an algebraic point of view this means that many simulators will provide an approximate solution by solving a linear system of algebraic equations characterized by an ill conditioned matrix. Unfortunately, the particular dielectric configuration and the particular working frequency allowing the existence of a \( \text{TM}_{mn} \) “surface wave” satisfying Maxwell’s equations together with the boundary conditions on \( \Gamma_{pec} \) and \( \Gamma_{int} \) are less predictable and make more difficult any a priori estimate of cases which can be reliably managed by numerical simulators, especially when more challenging problems than that shown in Figure 1 are considered. Analogous considerations apply to the particular dielectric configuration and the particular working frequency allowing the existence of a \( \text{TE}_{mn} \) “surface wave”. 
The above considerations apply in particular to numerical simulators based on the finite element method and, according to [33], point out a limitation on the so-called “robustness” (i.e., “the property that the method is applicable for a large class of problems without any change” [33]) of (frequency domain) finite element simulators.

In order to provide some examples of the effects we have here analytically predicted we consider a geometric configuration as reported in Figure 1 with \( a = 0.02 \) m, \( b = 0.01 \) m, \( z_1 = -0.03 \) m, \( z_2 = 0.02 \) m. In all considered cases the working frequency is \( f = 10 \) GHz. All problems are approximatively solved by using a finite element simulator implemented in a standard way (i.e., based on Galerkin’s method [15], [8], with first order edge elements [15], [8] on triangulations of the domain made up of tetrahedra). The surface integrals to take account of \( \mathbf{f}_{R1} \) (or \( \mathbf{f}_{R2} = 0 \)) [15], [13] are approximated by reading the values of \( \mathbf{f}_{R1} \) (or \( \mathbf{f}_{R2} = 0 \)) itself on the geometric centers of the triangular faces on \( \Gamma_1 \) (\( \Gamma_2 \)) of the elements of the mesh and by assuming that this vector field is constant on every triangular face discretizing \( \Gamma_1 \) (\( \Gamma_2 \)).

The mesh we consider in all our simulations is obtained by discretizing the volume of the waveguide with small cubes which in turn are divided into six tetrahedra. The number of small cubes is defined by discretizing the side along the \( x \), \( y \) and \( z \) axes with, respectively, 8, 4 and 20 equal segments. One should note that the mesh so defined is not so coarse, having, in all considered cases, more than ten elements per minimum wavelength.

In Figure 2 the amplitudes of the components of the electric field calculated by the finite element simulator are shown when \( \varepsilon_1 = 1.0, \mu_1 = 1.0, \varepsilon_2 = -1.0 \) and \( \mu_2 = -1.0 \). All field values are calculated along a uniformly spaced lines of 220 points. The line is parallel to the \( z \) axis. In particular, the coordinates of the points are given by \((x = 0.01005, y = 0.00505, z)\). A slight shift of the line of points with respect to the central line \((x = 0.01, y = 0.005, z)\) is considered in order to avoid that all points were placed on faces or edges of the mesh where the edge element solution can present a discontinuity (even though on some points this could still happen). The effects of the ill posedness of the continuous problem are evident. As a matter of fact, the right amplitudes are \([E_y]\) = 1 [V/m], \([E_x]\) = \([E_z]\) = 0 but the amplitudes calculated by our finite element simulator are all well above 10 KV/m near the interface.

Another critical case for the numerical simulator, even though not pathological for the continuous problem, is obtained when \( \varepsilon_1 = 1.0, \mu_1 = 1.0, \varepsilon_2 = -1.11 \) and \( \mu_2 = -0.522722205 \) since condition (19) is almost exactly satisfied for \( f = 10 \) GHz, \( m = 1 \) and \( n = 1 \) (TM_{11} has cutoff frequencies above 15 GHz). The amplitudes of the different components of the electric field calculated by the finite element simulator are shown in Figure 3. The results are unreliable, even though the quality of the approximation is better than that obtained above.

In order to remove any doubt on the usual performances of our simulator we consider two other less critical examples. In the first of these \( \varepsilon_1 = 1.0, \mu_1 = 1.0, \varepsilon_2 = 1.69 \) and \( \mu_2 = 1.0 \). The amplitudes of the components of the electric field calculated along the same line of points as before by the finite element simulator are shown in Figure 4. The obtained approximation is very good.

The second less critical example is obtained by considering \( \varepsilon_1 = 1.0, \mu_1 = 1.0, \varepsilon_2 = -1.0 - j0.1 \) and \( \mu_2 = -0.67 \). The values of the dielectric parameters are such that condition (19) is not satisfied at the considered frequency for any \( m > 0 \) and \( n > 0 \). In Figure 5 the amplitudes of the components of the electric field calculated by the finite element simulator along the same line of points as before are shown. The quality of the approximation is not so good as in the case with only double positive media but the results can still be considered as satisfactory even though a non negligible error is present near the interface.

Another important byproduct of our analysis is that the performances of the iterative solvers usually exploited by finite element simulators [15] can be much worse than usual when double negative metamaterials are involved. As a matter of fact, it is well known that the performances of iterative solvers [34] are heavily affected by the condition number of the matrix of the system they have to solve. This consideration applies in particular to the biconjugate gradient method [15], which is very widely used in finite element simulators. This prediction is confirmed by our simulations. We have always used an algebraic solver based on the biconjugate gradient method (without any preconditioning). In all performed simulations we had to deal with 3864 degrees
Figure 2: Amplitude of the electric field components calculated by the finite element simulator along a line parallel to the waveguide axis. “Conjugate” double positive media and double negative metamaterials are involved.

Figure 3: Amplitude of the electric field components calculated by the finite element simulator along a line parallel to the waveguide axis. The involved double positive media and double negative metamaterials are characterized by dielectric parameters that satisfy, for $f = 10$ GHz, $m = 1$ and $n = 1$, almost exactly condition (19).
Figure 4: Amplitude of the electric field components calculated by the finite element simulator along a line parallel to the waveguide axis. Only double positive media are involved.

Figure 5: Amplitude of the electric field components calculated by the finite element simulator along a line parallel to the waveguide axis. Double positive media and double negative metamaterials are involved but the values of the dielectric parameters are not critical.
of freedom and the number of iterations was 1301 in the case with only double positive media, 22599 in the less critical case involving double positive and double negative media, 30403 when the dielectric parameters of the double positive and double negative media involved satisfy condition (19) almost exactly but the continuous problem is well posed and 840970 when conjugate double positive and double negative media are involved.

These considerations do not affect our previous conclusions on the precision of the calculated results. As a matter of fact, for the most critical case reported above the same results were also obtained by using a method based on the $LDL^T$ decomposition [15].

Many of the practical aspects here considered could be further investigated for example by considering different dielectric values, different meshes, or different algebraic solvers. However, this kind of considerations would require the development of another work. Nonetheless, it could be important to note that some results on the precision of finite element simulators and on the performances of iterative solvers when double negative metamaterials are involved have been obtained in two previous works [16], [17] of the author. However, the effects observed in those manuscripts were by far less evident. Moreover a physical and mathematical interpretation of the observed effects was missing. On the contrary, it is believed that the theoretical results we have here deduced could be extrapolated to provide an interpretation of the anomalous performances of finite element simulators also in other more complicated contexts, such as that considered in [17]. The present study was actually motivated by the necessity to explain these anomalous performances.

Finally, our theoretical results could also provide the essential motivation for the anomalous performances of commercial finite element simulators pointed out in [18] and [19]. As a matter of fact, even though these analyses are devoted to the study of other topics, the authors have observed that the field at the air/metamaterial interface can be very strong and that extremely dense meshes can be necessary in order to obtain accurate results.

5 Conclusions

A set of very simple waveguide discontinuity problems have been considered. From their study it is shown, for the first time to the best of author’s knowledge, that infinite time-harmonic electromagnetic driven problems can be ill posed even in the presence of absorbing boundary conditions on part of the boundary. The theoretical analysis is used to shed more light on the reliability of models presenting sharp interfaces between lossless double positive and double negative materials and to explain the anomalous results recently obtained by using finite element based numerical simulators when double negative metamaterials are involved. Furthermore, a couple of numerical examples provide a confirmation of our theoretical predictions.

References


An ill posed waveguide discontinuity problem involving metamaterials with...


