A theoretical warning for users of frequency-domain numerical simulators in the presence of metamaterials

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April 19, 2007

Abstract

In this paper it is shown that users of frequency-domain numerical simulators should be warned of the danger of using models with adjacent complementary media since such a configuration of materials can be a cause of a numerical instability which cannot be overcome by the presence of other lossy media or absorbing boundary conditions. The physical realizability at any frequency of models considering complementary media with sharp interfaces is then questioned by the contradiction between the deduced result and the one which holds true when the actual dielectric structure of metamaterials is considered.

1 Introduction

Nowadays electromagnetic numerical simulators are very popular and they are widely exploited even from a commercial point of view.

The popularity of these computer aided design tools is due to the fact that most of them are able to guarantee a priori several important features. As a matter of fact, most of them are sufficiently versatile to allow the simulation of a wide class of electromagnetic problems, and sufficiently efficient to provide the solution of even complicated problems in a reasonable amount of time. Most of commercial simulators are even user friendly. However, the reliability of a simulator is the only feature all users consider as inalienable. As a matter of fact, most of the above indicated features are useless if a simulator is not able to reproduce with a certain level of fidelity what happens in the models of interest.

From a practical point of view it is not possible to guarantee the reliability of a simulator in general, without hypotheses restricting the class of problems for which such a result can be obtained. Fortunately, very general results of reliability are available for some frequency-domain numerical methods [1] and the set of hypotheses under which these results are obtained is usually very simple. For example, if only double positive [2] and linear materials are involved, for simulators based on the finite element method it is sufficient that some losses are present in the dielectric media occupying just a subregion of the domain of numerical investigation or that absorbing boundary conditions are present on some part of the boundary [1]. As a consequence, if only double positive and linear materials are involved, a very basic knowledge of electromagnetics is sufficient to avoid the cases in which the method is not reliable and even non-expert users can obtain satisfactory results from electromagnetic numerical simulators in the design of complex electromagnetic systems.

The recent advent of metamaterials [3], [2], [4] has partially modified the previously depicted situation. As a matter of fact, metamaterials are studied by considering two approaches. In one of these all inhomogeneities are considered and, even though it usually results in extremely
complicated numerical problems, all previous considerations on the reliability of simulators apply since only double positive media are involved. On the contrary, with the second approach, based on the definition of effective constitutive parameters, the advantage usually obtained in terms of complexity of the numerical problems to be dealt with is paid with the lack of very general results of reliability. There are, actually, some results in this direction for frequency-domain simulators based on the finite element method [5] but they are obtained under hypotheses which are not sufficiently general to cover cases of interest such as those studied in [6] or [7]. On the contrary, there are some recent experimental results [8], [9], [10], [11], which suggest that the reliability and the efficiency of time-harmonic finite element based numerical simulators could be questioned. However, even a large number of experiments cannot be used to deduce that the simulator considered is not able to provide converging sequences of approximations. A much stronger indication that the situation could be less satisfactory when metamaterials described as effective media are present than in cases involving just double positive media can be deduced from [12]. In that paper some time-harmonic electromagnetic boundary value problems involving complementary (or conjugate) media [13] are theoretically shown to be ill-posed [12] and, as a by-product, it is shown that they cannot be reliably simulated by most numerical simulators. However, there could still be some hope to obtain sufficiently general and simple reliability results for frequency-domain numerical simulators since the negative results in [12] have been obtained by considering only lossless double positive and double negative media, with matched impedance boundary conditions on the two ports, meaning that both materials are thought of as effective homogeneous media of infinite extensions. These hypotheses could be considered as unreasonably idealistic.

This paper shows that our hope to retain the validity of simple and physically sound results of reliability, as the one reported above concerning the sufficiency of the presence of some losses in part of the media or in part of the boundary for the reliability of finite element based simulators, is vain. In particular we show that, without assuming the critical hypotheses of [12] mentioned above, it is possible to obtain the same result, that is the ill-posedness of simple waveguide discontinuity problems which has so terrible effects on numerical simulators. The main consequence of the above considerations is that users of frequency-domain numerical simulators have to be warned about the fact that the presence of adjacent complementary media can be a very dangerous cause of numerical instabilities and that, unfortunately, such instability cannot be overcome, contrary to our basic physical feeling, by the presence of any kind of losses.

A by-product of our analysis concerns the physical realizability of models considering complementary media with sharp interfaces. In particular, our results show that by considering inclusions of double positive media it is not possible to obtain two adjacent conjugate media in a waveguide as those here considered. This consideration enforces the analogous one anticipated in [12] since the models here considered are more realistic than those analyzed in [12].

This paper is organized as follows. In Section 2 the simple waveguide discontinuity problems considered in this paper are defined. The idea exploited in this paper to prove that such problems are ill posed even in the presence of material losses is shown in Section 3, even though most analytical details are shown in Appendix A. Finally, in Section 4 a discussion on the practical consequences of the theoretical results presented in this paper is provided.

## 2 Formulation of simple waveguide discontinuity problems

The geometry of the simple waveguide discontinuity problems considered in this paper is reported in Figure 1 along with the cartesian reference system. In a rectangular waveguide of section $a \times b$ a discontinuity due to two layers of different materials is present. The first of the layers making up the discontinuity occupies region 1, identified by $\Omega_1 = \{(x, y, z) \in \mathbb{R}^3 : x \in (0, a), y \in (0, b), z \in (z_1, 0)\}$. The homogeneous medium in this region is characterized by $\varepsilon_1 > 0$ and $\mu_1 > 0$. The second layer of the discontinuity is made up of a so-called double negative materials [2] and occupies region 2, defined by $\Omega_2 = \{(x, y, z) \in \mathbb{R}^3 : x \in (0, a), y \in (0, b), z \in (0, z_2)\}$. Its constitutive parameters are assumed to be $\varepsilon_2 < 0$ and $\mu_2 < 0$. The rest of the numerical domain is denoted by $\Omega_3$. In this region the linear and isotropic material is characterized by $\varepsilon_3 \in \mathbb{C}$ and
permeability $\mu_3 \in \mathbb{C}$. It is a traditional passive medium with $Re(\varepsilon_3) > 0$, $Re(\mu_3) > 0$, $Im(\varepsilon_3) \leq 0$ and $Im(\mu_3) \leq 0$. Such a material can be lossy without restrictions.

According to [14] (section 8.5, pp. 263-267) it is assumed that the waveguide walls, denoted by $\Gamma_{\text{pec}}$, are made up of a perfect electric conductor and on the two ports (at $z = z_3$ and $z = z_4$) denoted by $\Gamma_1$ and $\Gamma_2$, possibly inhomogeneous impedance boundary conditions are enforced. Finally, $\Gamma_{\text{int},1}$, $\Gamma_{\text{int},0}$ and $\Gamma_{\text{int},2}$ denote the interface between different materials respectively at $z = z_1$, $z = 0$ and $z = z_2$.

By taking account of the possible presence of impressed sources in the waveguide [15] (p. 281), the mathematical formulation of the problem of interest can be the following

$$\begin{align*}
\nabla \times \mathbf{H} - j\omega\varepsilon\mathbf{E} &= \mathbf{J} & \text{in } \Omega_i, & i = 1, 2, 3 \\
\nabla \times \mathbf{E} + j\omega\mu\mathbf{H} &= 0 & \text{in } \Omega_i, & i = 1, 2, 3 \\
\mathbf{E} \times \mathbf{n} &= 0 & \text{on } \Gamma_{\text{pec}} \\
\mathbf{E} \big|_{z=z_1^-} \times \hat{z} &= \mathbf{E} \big|_{z=z_1^+} \times \hat{z} & \text{on } \Gamma_{\text{int},1} \\
\mathbf{H} \big|_{z=z_1^-} \times \hat{z} &= \mathbf{H} \big|_{z=z_1^+} \times \hat{z} & \text{on } \Gamma_{\text{int},1} \\
\mathbf{E} \big|_{z=0^-} \times \hat{z} &= \mathbf{E} \big|_{z=0^+} \times \hat{z} & \text{on } \Gamma_{\text{int},0} \\
\mathbf{H} \big|_{z=0^-} \times \hat{z} &= \mathbf{H} \big|_{z=0^+} \times \hat{z} & \text{on } \Gamma_{\text{int},0} \\
\mathbf{E} \big|_{z=z_2^-} \times \hat{z} &= \mathbf{E} \big|_{z=z_2^+} \times \hat{z} & \text{on } \Gamma_{\text{int},2} \\
\mathbf{H} \big|_{z=z_2^-} \times \hat{z} &= \mathbf{H} \big|_{z=z_2^+} \times \hat{z} & \text{on } \Gamma_{\text{int},2} \\
\mathbf{H} \times \mathbf{n} - Y_c(\mathbf{n} \times \mathbf{E} \times \mathbf{n}) &= \mathbf{f}_{r_i} & \text{on } \Gamma_i, & i = 1, 2
\end{align*}$$

(1)

where $\hat{z}$ is the coordinate versor along the $z$ axis, $\mathbf{n}$ is the unit vector normal to the boundary ($\Gamma_{\text{pec}}, \Gamma_1$ or $\Gamma_2$), $Y_c \in \mathbb{C}$ is the boundary admittance which defines absorbing boundary conditions whenever $Re(Y_c) > 0$ and $\mathbf{f}_{r_i}, i = 1, 2$, are the known terms modeling the incident waves on the two ports [14]. Usually $Y_c$ and $\mathbf{f}_{r_i}, i = 1, 2$, are defined in order to take account of the TE$_{10}$ mode incidency.

3 Behaviour of the considered models

It has been recently shown [12] that the presence of a simple discontinuity made up of conjugate media can give unexpected results. In particular, in [12] it is shown that if $\Omega_1$ (see Figure 1) is not present and the regions 1 and 2 are bounded by two ports at $z = z_1$ and $z = z_2$ matched for the propagation of the TE$_{10}$ mode (meaning that these regions are thought of as having infinite longitudinal extension) the corresponding time-harmonic electromagnetic driven problem is ill posed in the sense that the solution does not depend continuously on the data (the source terms appearing in impedance boundary conditions). However, as already pointed out, the infinite
longitudinal extension of the two materials involved and the absence of material losses could be considered as unrealistic.

These hypotheses, however, are shown in this section to be unnecessary. Against our physical feeling, we show that even in the presence of lossy media and absorbing boundary conditions the models considered in Section 2 can be ill posed.

The ill posedness is again due to the fact that the solution does not depend continuously on the data. Even though all analytical details are provided in Appendix A, in this section we show in which way the idea exploited in [12] can be modified to take account of the more realistic situation of interest. For the reader convenience we briefly recall the basic technical reasons behind the result shown in [12].

In the presence of only complementary media of infinite extension, infinite many electromagnetic fields decaying exponentially as one moves away from the interface are solutions of Maxwell’s equations and satisfy the right continuity conditions at the interface between the two media involved. These fields are given by the TE$_{mn}$ or TM$_{mn}$ modes of the rectangular waveguide. They are all below cutoff at the working frequency and present a dependence of all the field components on the longitudinal coordinate given by

$$e^{\gamma_{mn} z}, \quad z < 0$$

and

$$e^{-\gamma_{mn} z}, \quad z > 0,$$

where

$$\gamma_{mn} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} - \omega^2 \epsilon_1 \mu_1$$

$$= \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} - \omega^2 \epsilon_2 \mu_2 > 0.$$  

It is important to note that the exponential decay becomes more and more important as $m$ or $n$ (and then $\gamma_{mn}$) becomes bigger and bigger.

In [12] it is shown that the electromagnetic driven problem admits a unique solution and all these modes below cutoff are considered as variations of the solution (by superposition with the actual solution). The usual $L^2$ norm of these variations, i.e., the $L^2$ norm of the electric and magnetic fields of these modes, given by

$$\int_{\Omega_1} (\mathbf{E} \cdot \mathbf{E}^* + \mathbf{H} \cdot \mathbf{H}^*) \, dV + \int_{\Omega_2} (\mathbf{E} \cdot \mathbf{E}^* + \mathbf{H} \cdot \mathbf{H}^*) \, dV,$$

necessarily presents the following term in the first addend

$$\int_{z_1}^0 e^{2\gamma_{mn} z} \, dz = \frac{1}{2\gamma_{mn}} (1 - e^{2\gamma_{mn} z_1})$$

and the following term in the second one

$$\int_{z_2}^0 e^{-2\gamma_{mn} z} \, dz = \frac{1}{2\gamma_{mn}} (1 - e^{-2\gamma_{mn} z_2}).$$

Both decay as $\frac{1}{\gamma_{mn}}$ as $m$ or $n$ (and, then, $\gamma_{mn}$ itself) becomes large.

On the contrary, the $L^2$ norm of the variation of the inhomogeneous terms present in the impedance boundary conditions to be enforced in [12] at $z = z_1$ and $z = z_2$, given by

$$\int_{\Gamma_1} f_{r1mn} \cdot \mathbf{r}_{1mn} \, dS + \int_{\Gamma_2} f_{r2mn} \cdot \mathbf{r}_{2mn} \, dS,$$
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necessarily presents two terms proportional to $e^{2\gamma_m z_1}$ and $e^{-2\gamma_m z_2}$, respectively.

Now the conclusion in [12] easily follows since the $(L^2$ norm of the) variations of the source terms decay exponentially with $\gamma_m$ and then with $m$ or $n$ whereas the $(L^2$ norm of the) variations of the fields decays as $\frac{1}{\gamma_m}$.

The above consideration can be modified as follows to take account of the finite longitudinal extension of the media making up the discontinuity and then of the presence of the medium in the region $\Omega_3$ (see Figure 1). In order to simplify our deduction, without reducing the generality of the result, we can restrict the set of modes to the TE$_{m0}$ ones.

At the beginning we consider in the two complementary media exactly the same fields we considered in [12]. Due to the increasing exponential decay, away from the interface at $z = 0$ these modes have smaller and smaller amplitudes as $m$ becomes large ($n = 0$). Thus, in the regions $z \in (z_1, \frac{z_2}{2})$ and $z \in (\frac{z_2}{2}, z_2)$ we can "slightly" modify the $y$ component of these TE$_{m0}$ modes (the only component of the electric field $\mathbf{E}_{m0}$ of these modes) in order to obtain a field $\mathbf{E}_{m0}$ which is continuous with continuous derivatives and such that $\mathbf{E}_{m0} = 0$ in the regions $z \leq z_1$ and $z \geq z_2$. Just the dependence of the original field on the $z$ coordinate is affected.

In the regions where we modify the electric field of the modes we define accordingly

$$\mathbf{H}_{m0} = \frac{j}{\omega \mu} \nabla \times \mathbf{E}_{m0}$$

(9)

and

$$\mathbf{J}_{m0} = \nabla \times \mathbf{H}_{m0} - j\omega \varepsilon \mathbf{E}_{m0}, \quad i = 1, 2, 3$$

(10)

to retain the validity of Maxwell’s curl equations. In Appendix A it will be shown that all equations in (1) are satisfied by these fields. In particular, the last equation in equation (1) is satisfied with $f_i = 0$, $i = 1, 2$, and the source term is just given by $\mathbf{J}_{m0}$.

All modifications introduced on the $L^2$ norms of $\mathbf{E}_{m0}$ and $\mathbf{H}_{m0}$ are small because these changes have been done in regions where the original modes have already been exponentially attenuated. Moreover, these variations become exponentially smaller and smaller as $m$ becomes larger and larger. Thus the $L^2$ norms of these fields decay as $\frac{1}{\gamma_m}$ exactly as the original modes.

For the same reason, the $L^2$ norm of the source term $\mathbf{J}_{m0}$ are small and becomes exponentially smaller and smaller as $m$ becomes larger and larger due to the increasing value of $\gamma_{m0} > 0$ and then of the exponential decay.

In summary, as $m$ becomes larger and larger, variations of the source terms which decay exponentially with $\gamma_{m0}$ give variations of the solutions which decay as $\frac{1}{\gamma_{m0}}$ and, as $m \to \infty$, there is no way to control the amplitude of the variation of the solution with the amplitude of the variation of the source terms. For these basic reasons also these more general problems are ill posed.

4 Theoretical and practical consequences

The previous deduction shows that a time-harmonic electromagnetic driven problem can be ill posed when adjacent complementary media are involved. This can happen at any frequency, independently of the presence of material losses and absorbing boundary conditions and independently of the thickness ($|z_1|$ and $z_2$) of the two complementary media. This consideration, together with the fact that the fields defined above are trivial except for a region close to the interface between adjacent media, allows us to deduce that the same conclusion can be easily obtained for discontinuities made up of an arbitrary number of layers or even of materials with different geometries, provided that an interface between complementary media on a plane $z = \text{constant}$ is present.

The most important practical consequence of this result is related to the fact that most frequency-domain numerical simulators are based on formulations like (1) of the problems of interest or on equivalent formulations. All numerical techniques introduce some kind of errors in the discretization process and, when the solution of the discretized problem does not depend
continuously on the data, these errors, independently of their amplitudes, are sufficient to introduce very large deviations of the results from the actual solution. Thus such frequency-domain numerical simulators are unreliable when they are exploited to approximate the solution of an ill posed problem and, unfortunately, such unreliability cannot be overcome by considering finer discretizations.

The described situation is completely different from the common experience and, unfortunately, the conditions that are usually sufficient to guarantee the numerical reliability are not sufficient anymore for the same conclusion to hold true. As a matter of fact, in the presence of only double positive media, either types of losses indicated above are sufficient, independently of the presence of each other, to guarantee the well posedness of time-harmonic electromagnetic driven problems [1] and, for example, the reliability of numerical simulators based on the finite element method [1].

Thus, in practice, it could be important to warn users of the possibility of obtaining totally unreliable numerical results when frequency-domain numerical simulators are exploited for the design of new components involving metamaterials (dealt with as effective media). In this sense our conclusion is in agreement with the considerations reported in [16] which suggest, in the final part of a design procedure, the use of numerical simulations which take account of the actual dielectric structure of the metamaterials involved.

As an example of what can happen, let us consider the problem characterized by $z_1 = -0.03$, $z_2 = 0.03$, $z_3 = -0.06$, $z_4 = 0.06$, $a = 0.02$, $b = 0.01$ (all lengths in meters), $f = 9 \, \text{GHz}$, $\varepsilon_1 = \varepsilon_0$, $\mu_1 = \mu_0$, $\varepsilon_2 = -\varepsilon_1$, $\mu_2 = -\mu_1$, $\varepsilon_3 = (1.0 - j0.1) \varepsilon_0$ and $\mu_3 = (1.0 - j0.1) \mu_0$. A $\text{TE}_{10}$ mode (which, by the way, is above cutoff) of unit amplitude is considered as the incident field on $\Gamma_1$ and the other port is matched ($Y_e = \frac{9 \, \text{GHz}}{\omega \varepsilon_0}$, $r = \sqrt{\omega^2 \varepsilon_0 \mu_0}$, $f_1 = 2 \sqrt{\frac{2}{\pi b}} Y_e \sin \left( \frac{\pi x}{a} \right) e^{-jz_3 \sqrt{\omega^2 \varepsilon_0 \mu_0 - \frac{b^2}{a^2}}} \hat{y}$ and $f_2 = 0$). The solution is approximated by using a finite element simulator based on Galerkin’s method [1], [14], with first order edge elements [14] on triangulations of the domain made up of tetrahedra. In Figure 2 the analytical solution and the numerical one are shown. The finite element approximation is, as expected, totally unreliable. According to our considerations the errors are very significant near the interface between conjugate media. The numerical approximation has been obtained by discretizing the domain with small cubes (2 mm on a side) which in turn are divided into six tetrahedra. It is important to recall that with the mesh considered the results would be satisfactory in the presence of only traditional media having more or less the same density.

A byproduct of our analysis, partially anticipated in [12] with the limitations reported in the Introduction, is that double positive materials cannot produce, at any frequency, a dielectric configuration with a sharp interface between complementary media on a transverse plane of a rectangular waveguide. As a matter of fact, the electromagnetic problem should be well posed if the fine structure of the double positive inclusions was considered and, at the same time, ill posed if these inclusions were able to generate the dielectric configuration just described.

This deduction supports from another perspective the consideration reported in [17] that models with sharp interfaces between double positive medium and homogeneous metamaterials should not be considered as representative of the dielectric structures exploited in practice to realize metamaterials in the middle of ordinary media.

5 Conclusions

The study of a set of almost trivial waveguide discontinuity problems has been carried out in order to show that users of frequency-domain numerical simulators should be warned of the danger of using models with adjacent complementary media. Such a configuration of effective materials has been shown to be the cause of a numerical instability which cannot be overcome by the presence of any kind of losses. It is then shown by contradiction that all effective models considering complementary media with sharp interfaces cannot be representative of what is obtained in practice with the use of double positive inclusions in double positive host media.
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Figure 2: Amplitude of $E_y$ calculated along a line close and parallel to the waveguide axis. The analytical solution is compared with the one obtained by using a finite element simulator.

A Appendix

It is easy to verify that if in the waveguide we had just two complementary media with the dielectric interface at $z = 0$ the following electric and magnetic fields of a TE$_{m0}$ mode would satisfy Maxwell equations and the continuity of the tangential components at the interface

$$E_{m0} = -\hat{y} E_{ym0}(x) f(z) = -\hat{y} \sqrt{\frac{2}{ab}} V_m \sin \left(\frac{m\pi x}{a}\right) f(z)$$  \hspace{1cm} (11)

$$H_{m0} = \frac{j}{\omega \mu_i} \nabla \times E_{m0} = \frac{j}{\omega \mu_i} \left( \hat{x} E_{ym0}(x) \frac{d}{dz} f(z) - \hat{z} f(z) \frac{d}{dx} E_{ym0}(x) \right)$$  \hspace{1cm} (12)

$$= \frac{j}{\omega \mu_i} \sqrt{\frac{2}{ab}} V_m \left( \hat{x} \sin \left(\frac{m\pi x}{a}\right) (-1)^{i+1} \gamma_{m0} f(z) - \hat{z} \left( \frac{m\pi}{a} \right) \cos \left(\frac{m\pi x}{a}\right) f(z) \right)$$

with $i = 1$ if $z < 0$, $i = 2$ if $z > 0$ and

$$f(z) = \begin{cases} e^{\gamma_{m0} z} & \text{for } z < 0 \\ e^{-\gamma_{m0} z} & \text{for } z > 0. \end{cases}$$  \hspace{1cm} (13)

$V_m$ is an arbitrarily defined complex constant.

As we have already pointed out in Section 3, we modify this field at some distance from the interface. In particular we consider

$$\bar{E}_{m0} = -\hat{y} E_{ym0}(x) g(z) = -\hat{y} \sqrt{\frac{2}{ab}} V_m \sin \left(\frac{m\pi x}{a}\right) g(z)$$  \hspace{1cm} (14)

where

$$g(z) = \begin{cases} e^{\gamma_{m0} z} & \text{for } z \in \left(\frac{z_1}{2}, 0\right) \\ e^{-\gamma_{m0} z} & \text{for } z \in \left(0, \frac{z_3}{2}\right) \\ 0 & \text{for } z \in \left(z_3, z_1\right) \cup \left(z_2, z_4\right) \\ a_{m1} (z - z_1)^3 + a_{m2} (z - z_1)^2 & \text{for } z \in \left(z_1, \frac{z_3}{2}\right) \\ a_{m3} (z - z_2)^3 + a_{m4} (z - z_2)^2 & \text{for } z \in \left(\frac{z_3}{2}, z_2\right). \end{cases}$$  \hspace{1cm} (15)
For all \( z \in (\frac{n}{2}, \frac{n+1}{2}) \) we have \( g(z) = f(z) \).

The coefficients \( a_{mi} \), \( i = 1, \ldots, 4 \) are defined in order to guarantee the \( C^1 \) continuity of \( g(z) \) on the two sides of the interface at \( z = 0 \). There are two joints at \( z = z_1 \) and \( z = z_1/2 \) (\( z = z_2 \) and \( z = z_2/2 \)) on the left (right) side but the \( C^1 \) continuity at \( z = z_1 \) (\( z = z_2 \)) is automatically satisfied. These four unknown coefficients can be easily found by enforcing the \( C^1 \) continuity of \( g(z) \) at \( z = z_1/2 \) and \( z = z_2/2 \). They are given by

\[
\begin{align*}
a_{m1} &= \frac{4\gamma_m z_1 + 16}{z_1^2} e^{\gamma_m z_1^2/2} \\
a_{m2} &= \frac{2\gamma_m z_1 + 12}{z_1^2} e^{\gamma_m z_1^2/2} \\
a_{m3} &= \frac{16 - 4\gamma_m z_2}{z_2^2} e^{-\gamma_m z_2^2/2} \\
a_{m4} &= \frac{12 - 2\gamma_m z_2}{z_2^2} e^{-\gamma_m z_2^2/2}.
\end{align*}
\]

The third equation in equation (1) is satisfied by \( \mathbf{H}_{m0} \) since its dependence on the transverse coordinates is the same as that of the original \( \mathbf{E}_{m0} \) field. Moreover, the fourth and eighth equations in equation (1) are satisfied since \( \mathbf{E}_{m0} \) is continuous on \((z_1, 0)\) and \((0, z_4)\) and it is trivial at \( z = z_1 \) and \( z = z_2 \). Finally, the sixth equation in equation (1) holds true because it was satisfied by the original field and the new one is exactly the same for \( z \in (z_1/2, z_2/2) \).

Consistently with the approach indicated, we define the magnetic field \( \mathbf{H}_{m0} \) by enforcing the second equation in equation (1). Thus

\[
\mathbf{H}_{m0} = \frac{j}{\omega \mu_i} \nabla \times \mathbf{E}_{m0} = \frac{j}{\omega \mu_i} \left( \mathbf{x} E_{ym0}(x) \frac{d}{dz} g(z) - \mathbf{z} g(z) \frac{d}{dx} E_{ym0}(x) \right)
\]

\[
= \frac{j}{\omega \mu_i} \sqrt{\frac{2}{ab}} V_m \left( \mathbf{x} \sin \left( \frac{m\pi x}{a} \right) \frac{d}{dz} g(z) - \mathbf{z} \left( \frac{m\pi x}{a} \right) \cos \left( \frac{m\pi x}{a} \right) g(z) \right)
\]

where \( i = 1 \) if \( z \in (z_1, 0) \), \( i = 2 \) if \( z \in (0, z_2) \) and \( i = 3 \) if \( z \in (z_3, z_1) \) or \( z \in (z_2, z_4) \).

This field satisfies the fifth, seventh and ninth equations in equation (1), because it is continuous and trivial at \( z = z_1 \) and \( z = z_2 \) and because it is exactly the same as the original one around the interface at \( z = 0 \). Moreover, \( \mathbf{E}_{m0} \) and \( \mathbf{H}_{m0} \) satisfy the tenth equation in equation (1) with \( \mathbf{J}_1 = \mathbf{J}_2 = 0 \) since both fields are trivial in \( \Omega_3 \). Finally, to retain the validity of the first equation in equation (1) we define, with the same meaning of the subscript \( i \),

\[
\mathbf{J}_{m0} = \nabla \times \mathbf{H}_{m0} - j\omega\varepsilon_i \mathbf{E}_{m0}
\]

\[
= \frac{j}{\omega \mu_i} \hat{y} \left( E_{ym0}(x) \frac{d^2}{dz^2} g(z) + g(z) \frac{d^2}{dx^2} E_{ym0}(x) \right) + j \omega \varepsilon_i \hat{y} E_{ym0}(x) g(z)
\]

\[
= \frac{j}{\omega \mu_i} \hat{y} E_{ym0}(x) \left( \frac{d^2}{dz^2} g(z) - g(z) \left( \frac{m\pi x}{a} \right)^2 - \omega^2 \mu_i \varepsilon_i \right)
\]

It is trivial to verify that \( \mathbf{J}_{m0} = 0 \) for \( z \in (\frac{n}{2}, 0) \cup (0, \frac{n}{2}) \cup (z_3, z_1) \cup (z_2, z_4) \) for all \( x \in (0, a) \). In the subregions of \( \Omega_1 \) and \( \Omega_2 \) characterized by \( z \in (z_1, \frac{n}{2}) \) and \( z \in (\frac{n}{2}, z_2) \) where the current density is not trivial the media are complementary and (see equation (4)) \( \left( \frac{m\pi x}{a} \right)^2 - \omega^2 \mu_i \varepsilon_i \right) = \gamma_{m0}^2 \) so that, by defining also \( h(z) = \frac{d^2}{dz^2} g(z) - \gamma_{m0}^2 g(z) \)

\[
\mathbf{J}_{m0} = \frac{j}{\omega} \hat{y} E_{ym0}(x) h(z)
\]

\[
= \begin{cases} 
\frac{1}{\mu_1} & \text{for } z \in (z_1, \frac{n}{2}) \\
\frac{1}{\mu_2} & \text{for } z \in (\frac{a}{2}, z_2).
\end{cases}
\]

It is now possible to calculate and compare the \( L^2 \) norms of the fields \( \mathbf{E}_{m0} \) and \( \mathbf{H}_{m0} \) and of the corresponding source terms \( \mathbf{J}_{m0} \).
We have that, on the one hand
\[
\sum_{i=1}^{3} \int_{\Omega_i} \left( \mathbf{E}_{m0} \cdot \mathbf{E}_{m0}^* + \mathbf{H}_{m0} \cdot \mathbf{H}_{m0}^* \right) dV \\
\geq \int_{\Omega_2} \left( \mathbf{E}_{m0} \cdot \mathbf{E}_{m0}^* \right) dV \geq \int_{z=0}^{z_a} |g(z)|^2 dz \int_{x=0}^{a} |E_{ym0}(x)|^2 dx \\
= \frac{2}{ab} |V_m|^2 \int_{z=0}^{z_a} e^{-2\gamma_{m0}z} dz \int_{x=0}^{a} \left( \sin \left( \frac{m \pi x}{a} \right) \right)^2 dx \\
= \frac{1}{b} |V_m|^2 \int_{z=0}^{z_a} e^{-2\gamma_{m0}z} dz = \frac{1}{2\gamma_{m0} b} |V_m|^2 (1 - e^{-\gamma_{m0}z})
\]
which behaves like \(\frac{1}{\gamma_{m0}}\) as \(m\) and then \(\gamma_{m0}\) becomes larger and larger. On the other hand,
\[
\sum_{i=1}^{3} \int_{\Omega_i} \mathbf{J}_{m0} \cdot \mathbf{J}_{m0}^* dV \\
= \frac{1}{\omega^2} \int_{0}^{a} |E_{ym0}(x)|^2 dx \cdot \left( \frac{1}{\mu_1} \int_{z_1}^{z_2} |h(z)|^2 dz + \frac{1}{\mu_2} \int_{\frac{z_2}{2}}^{\frac{z_2}{2}} |h(z)|^2 dz \right) \\
= \frac{|V_m|^2}{b\omega^2} \left( \frac{1}{\mu_1} \int_{z_1}^{z_2} |h(z)|^2 dz + \frac{1}{\mu_2} \int_{\frac{z_2}{2}}^{\frac{z_2}{2}} |h(z)|^2 dz \right).
\]
By taking account of expressions (16)-(19) and of the fact that the polynomial function \(h(z)\) has always real values (\(\gamma_{m0}\) and \(\omega_{m1}, i = 1, \ldots, 4\) are real) we obtain:
\[
\sum_{i=1}^{3} \int_{\Omega_i} \mathbf{J}_{m0} \cdot \mathbf{J}_{m0}^* dV = \frac{|V_m|^2}{b\omega^2} \cdot \left( \frac{1}{\mu_1} \int_{z_1}^{z_2} (h(z))^2 dz + \frac{1}{\mu_2} \int_{\frac{z_2}{2}}^{\frac{z_2}{2}} (h(z))^2 dz \right) \\
= \frac{|V_m|^2}{b\omega^2} \cdot \left( \frac{e^{\gamma_{m0}z_1}}{\mu_1^2} \int_{z_1}^{\frac{z_2}{2}} \left( 6 \frac{4\gamma_{m0}z_1 + 16}{z_1^3} (z - z_1) + 2 \frac{2\gamma_{m0}z_1 + 12}{z_1^2} (z - z_1)^3 - \frac{2\gamma_{m0}z_1 + 12}{z_1^2} (z - z_1)^3 \right)^2 dz \\
+ \frac{e^{-\gamma_{m0}z_2}}{\mu_2^2} \int_{\frac{z_2}{2}}^{\frac{z_2}{2}} \left( 6 \frac{16 - 4\gamma_{m0}z_2}{z_2^3} (z - z_2) + 2 \frac{12 - 2\gamma_{m0}z_2}{z_2^2} (z - z_2)^3 - \frac{12 - 2\gamma_{m0}z_2}{z_2^2} (z - z_2)^3 \right)^2 dz \right).
\]
The two integrand functions of the last member are just polynomials of the sixth degree in \(\gamma_{m0}\), as it is easy to check. Thus, as \(m\) and then \(\gamma_{m0}\) becomes larger and larger the behaviour of the \(L^2\) norm of the source \(\mathbf{J}_{m0}\) is dominated by the exponential decay of \(e^{\gamma_{m0}z_1}\) and \(e^{-\gamma_{m0}z_2}\) and there is no way to control the amplitude of the variations on the fields, whose behaviour is given by \(\frac{1}{\gamma_{m0}^2}\), with the amplitude of the variations on the source terms. By definition this means that the solution of the considered problem does not depend continuously on the data.

References


