On the performances of electromagnetic finite element simulators in the presence of three-dimensional double negative scatterers

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Abstract

The effects of the presence of double negative metamaterials on the accuracy of the results computed by finite element simulators are investigated. The performances of the iterative solver usually exploited by the same simulators are analysed, too. The outcome is that finite element simulators could require much finer meshes than usual when double negative metamaterials are involved, especially in regions containing and slightly surrounding the subregions occupied by double negative media. This effect is particularly important when the double negative metamaterials involved have not a very high magnetic or electric loss tangent. The effect on the most popular iterative solver is even more evident.

1 Introduction

The interest of the research community on metamaterials has reached a high level due to the potential results this technology could provide [1], [2], [3]. It is well documented by the numerous conferences and journal articles [2], [3] and the high number of national and international research projects [4], [5], [6] dedicated to such a topic.

In this period of tumultuous expansion of this research area numerical simulators allowing the modelling of metamaterials have become more and more important [7], [8], [9]. The capability of simulators to deal with metamaterials is even exploited from a commercial point of view [10].

Even though several numerical techniques have been used to simulate electromagnetic problems involving metamaterials, the necessary results on the convergence of the approximations [11] provided by the simulators themselves are lacking. This is true especially when the unusual effective dielectric parameters of metamaterials are modelled. At the same time, the unusual behaviour of metamaterials makes the necessity of results on the convergence of the approximations as the number of unknowns becomes larger and larger even more important.

For finite-elements-based simulators [12] such results have been recently obtained and cover most cases of interest involving metamaterials [13], [14]. Thus one of the most important features of finite element simulators is not affected by the presence of metamaterials and this could make the popularity of such kind of numerical technique even stronger.

Convergence results, however, are not the only aspect of practical interest as far as the performances of numerical simulators are concerned. As a matter of fact, from a practical point of view, error estimates could even be of more interest [11] since they provide an approximate indication of the number of unknowns to be used in order to obtain a given error [12]. Error estimates for

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finite element simulators are approximately known (in a more or less rigorous way) in the case of standard materials [12] but only a few numerical experiments have been carried out when metamaterials are involved. Moreover, most of these simulations have not been dedicated to such an analysis.

In [15], [16] the first results in this direction, to the best of authors’ knowledge, were provided. An exhaustive assessment of the performances of finite element simulators in the presence of metamaterials could require the solution of many different electromagnetic problems. One-, two- and three-dimensional problems are just examples of classes of problems of interest. Each class is then made up of several problems with variations in terms of geometry, source terms and dielectric properties. Due to the practical importance of 3D simulators in [15], [16] one- and two-dimensional finite element simulators were not considered. However, an edge-element-based 3D simulator was exploited just to solve a subset of one-dimensional and two-dimensional electromagnetic problems. The first outcome of the analysis carried out in [15], [16] is that in the presence of “double negative” (DNG) [2] metamaterials it could be sensible, in order to obtain reasonable error performances, to use finer meshes than those suggested by the simple rule of thumb requiring ten elements per wavelength, at least for “weak” DNG scatterers [16]. This finer discretization can be restricted to the regions occupied by metamaterials (or just slightly bigger). This outcome is in agreement with the conclusion, deduced in [17] and [18] as by-products of the main analyses concerning other topics, that the field at the air/metamaterial interface is very strong and that “an extremely dense mesh is needed to obtain accurate results” [17]. The second outcome concerns the impressive deterioration of the performances of the biconjugate gradient iterative solver, thus making finite element simulators (usually exploiting this solver [12]) much slower than usual in the presence of DNG metamaterials. It is finally pointed out that losses help to reduce the above indicated negative effects.

In this paper, for the practical reason indicated above, the same edge-element-based (first order) 3D simulator is used. However, the simple one-dimensional and two-dimensional electromagnetic problems considered in [15] and [16] are now replaced by a 3D problem, thus providing indications on the more usual situations requiring the exploitation of numerical simulators. The 3D problem considered is “canonical” since the analytical solutions [19] are necessary in order to deduce if the presence of a DNG metamaterial can reduce the precision of the finite element simulator considered.

The outcome of this analysis is that the negative effects pointed out in [15] and [16] in the case of 1D or 2D problems can be much worse for truly 3D problems.

This paper is organized as follows. In Section 2 the 3D electromagnetic scattering problem of interest is defined and, for the reader convenience, some details on the finite element simulator exploited are provided. After having defined the triangulations and the error figures in Section 3, in Section 4 the numerical results are compared with the analytical ones and the effects of DNG metamaterials on the precision of finite element simulators are discussed. Finally, in Section 5 the performances of the most popular iterative solver are considered.

2 Formulation of the problem and its finite element approximation

As already pointed out, in order to analyze the precision of 3D finite element simulators for scattering problems in the presence of metamaterials, we consider, as a canonical problem allowing the calculation of analytical solutions, a time-harmonic uniform plane wave which illuminates a homogeneous sphere, which can be made up of DNG or DPS media. Two homogeneous regions can be identified. In the “external” region the incident plane wave propagates in a DPS medium characterized by the effective complex dielectric permittivity \( \varepsilon_1 \) and the effective complex magnetic permeability \( \mu_1 \). The same complex quantities of the DPS or DNG homogeneous scatterer are denoted by \( \varepsilon_2 \) and \( \mu_2 \), respectively. The additional subscript \( r \) will be used to denote the corresponding relative quantities. In the following, \( \text{Re} \) and \( \text{Im} \) denote the real and the imaginary
parts, respectively.

The incident electric field is supposed to be given by $E_{inc} = xe^{-jk_1z}$, with $k_1 = \omega\sqrt{\mu_1\varepsilon_1}$ as usual.

As already pointed out the problem considered is numerically studied by a 3D finite element simulator based on a total field formulation [12]. Finite element methods are usually deduced by discretizing weak formulations of the electromagnetic problems of interest [12]. Since finite element algorithms cannot deal with unbounded domains [12] the weak formulation is given in a bounded region $\Omega$. In the following $\Omega$ will be assumed to be a cube containing the sphere (the center of the sphere is the center of gravity of $\Omega$).

Weak formulations require the definition of boundary conditions [12]. In this work we assume to know on the domain boundary $\Gamma$ the exact right-hand side $f_R$ of the boundary condition

$$H \times n - \xi(n \times E \times n) = f_R$$

together with $\xi$, which is the scalar complex admittance involved in impedance boundary conditions [12], [13]. This is possible since the analytic solution is available. In the case of interest even approximate Sommerfeld radiation conditions could be enforced by deducing $f_R$ from the incident field [12]. The reader should be aware, however, that our analysis on the accuracy of the results would be heavily influenced by the use of first order (Sommerfeld) absorbing boundary conditions [12]. As a matter of fact, they would introduce an additional source of errors and, moreover, since they should be enforced at a given distance from the scatterer in order to obtain a good approximation, they would require much more degrees of freedom localized in the less critical part (i.e. the host medium) in our finite element analyses.

According to our hypothesis the starting point of our formulation is

$$\begin{cases} 
\nabla \times H = j\omega\varepsilon E & \text{in } \Omega \\
\nabla \times E = -j\omega\mu H & \text{in } \Omega \\
H \times n - \xi(n \times E \times n) = f_R & \text{on } \Gamma
\end{cases}$$

For such a problem a variational formulation is introduced [12], [13], [20]. By using the usual scalar product in $(L^2(\Omega))^3$ ($\int_\Omega u \cdot v^* \, dV$, where $^*$ indicates the complex conjugate) the variational formulation of (2) is [12], [13], [20]

$$\text{find } E \in V = H_{L^2,\Gamma}^2(\text{curl}, \Omega) \text{ such that}$$

$$\int_\Omega \mu^{-1}(\nabla \times E) \cdot (\nabla \times v^*) \, dV - \omega^2 \int_\Omega \varepsilon E \cdot v^* \, dV$$

$$+ j\omega \int_\Gamma (\xi(n \times E \times n) \cdot (n \times v^* \times n)) \, dS$$

$$= -j\omega \int_\Gamma f_R \cdot (n \times v^* \times n) \, dS \quad \forall v \in V.$$  

Our finite element simulator is implemented in a standard way and is based on Galerkin’s method [12], [20]. First order edge elements [12], [20] on triangulations of $\Omega$ made up of tetrahedra are used, since they are the most common elements exploited in practice. The surface integral on the right-hand side of (3) is approximated by reading the values of $f_R$ on the centers of gravity of the triangular faces of the elements of the mesh on the boundary of the domain and by assuming that this vector field is constant on every triangular face on $\Gamma$.

In order to carry out the analysis of interest it is important to know in advance that the considered problem formulations ((2) or (3)) are well posed and that the finite element method provides converging approximations as the number of unknowns becomes larger and larger [11]. For the problem and the numerical method we are interested in such results are already available (see [20] for DPS scatterers and [13] or [14] for DNG scatterers). In order to exploit these results in all our simulations involving DNG metamaterials we will never consider media with an electric or a magnetic loss tangent lower than 0.01, which, by the way, is higher than the usual values of the loss tangent of DPS materials used in microwave technology.
3 Definitions and error figures

In all cases we consider, the incident plane wave has a frequency of 300 MHz and the radius of the spherical scatterer is equal to \( r = 0.2 \) m. The side length of the cubic domain \( \Omega \) containing the scatterer is \( sl = 0.5 \) m. All points in \( \Omega \) are such that \( x, y, z \in [-sl/2, sl/2] \).

As far as the discretization is considered, the cubic domain is always divided into \((n-1) \times (n-1) \times (n-1)\) identical cubes, each divided into six tetrahedra. \( n \) gives the number of uniformly spaced points per edge of \( \Omega \). The possible values of the parameter \( n \) considered in our simulations are: \( n = 11, 31, 51, 71 \). In Table 1 we report the different features of the meshes exploited (\( w \) represents the maximum side length [m] of the small cubes in which \( \Omega \) is divided).

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Table 1: Features of the triangulations exploited in the simulations

The reader should note that since the efficiency of the simulator is not under investigation, we have deliberately avoided any consideration about the possible non uniform meshes which could be exploited.

It is important to note that the finite element simulator considers scatterers having just approximate shapes (curvilinear elements are not considered in our simulations). As a matter of fact, the dielectric parameters of the scatterers \( \varepsilon_2 \) and \( \mu_2 \) are assigned to a given tetrahedron just if its center of gravity belongs to the region which should be filled by the scatterer. All other elements are characterized by the dielectric parameters \( \varepsilon_1 \) and \( \mu_1 \) of the external medium. The shape approximation of the spherical scatterer can be considered satisfactory for relatively high values of \( n \). For \( n = 11 \) the approximation can present irregularities as large as 5 cm in a sphere of 40 cm of diameter and for this reason we have always considered much finer meshes. The mesh with \( n = 11 \) is just considered to provide some simple preliminary indications on the behaviour of the simulator.

All field values are calculated along three uniformly spaced lines of 201 points. The lines are parallel to the cartesian axes. In particular, the coordinates of the points of the three lines are given by \((x, y = t, z = t)\) (in m), \((x = t, y, z = t)\) and \((x = t, y = t, z)\), \( t = 0.003 \) m. The values of \( t \) were chosen to avoid that all points were placed on faces or edges of the mesh where the edge element solution can present a discontinuity (even though on some points of the sequences this could still happen). \( t \neq 0 \) is also necessary to prevent the calculation of the analytical solutions on singular points in the spherical coordinate system.

In order to estimate the possible deterioration of the precision of finite element simulators when DNG materials are involved, let us consider different figures of merit. In the following we will calculate the absolute error

\[ e_a(s) = E_{\text{analytic}}(s) - E_{\text{fem}}(s), \quad s = x, y, z \]  

and the relative error

\[ e_r(s) = \frac{E_{\text{analytic}}(s) - E_{\text{fem}}(s)}{E_{\text{analytic}}(s)}, \quad s = x, y, z \]  

along the three lines of points indicated above, with the obvious meaning of the different symbols. The corresponding mean values will be evaluated as well

\[ E_{\text{ams}} = \frac{1}{s_{\text{max}} - s_{\text{min}}} \int_{s_{\text{min}}}^{s_{\text{max}}} |e_a(s)| \, ds, \quad s = x, y, z, \]  

\[ E_{\text{rms}} = \frac{1}{s_{\text{max}} - s_{\text{min}}} \int_{s_{\text{min}}}^{s_{\text{max}}} |e_r(s)| \, ds, \quad s = x, y, z. \]
4 Accuracy of the finite element solution

In order to deduce a complete information on the possible deterioration of the performances of finite element simulators in the presence of DNG metamaterials we should consider wide variations of $\Re(\varepsilon_{ri})$, $\Im(\varepsilon_{ri})$, $\Re(\mu_{ri})$, $\Im(\mu_{ri})$, $i = 1, 2$, and $n$.

Unfortunately, as a consequence of the deterioration of the performances of the iterative solvers usually exploited by finite element simulators (see Section 5 and [16]), such a “brute force” approach is extremely time consuming. For this reason, we divide our analysis into two parts and the focus will be on deriving some indications on what “can” happen to the performances of finite element simulators in the presence of DNG metamaterials, by considering a finite set of possible configurations of $\varepsilon_{ri}$, $\mu_{ri}$, $i = 1, 2$ and $n$ that, however, are representative of the problem at hand. Thus, in the first stage of our analysis we choose $\Re(\varepsilon_{r1}) = 1.0$, $\Re(\mu_{r1}) = 1.0$, $\Im(\varepsilon_{r1}) = -0.3$, $\Im(\mu_{r1}) = 0.0$, $\Im(\varepsilon_{r2}) = 0.0$, $\Im(\mu_{r2}) = -0.3$, $n = 31$ and we allow $\Re(\varepsilon_{r2})$ and $\Re(\mu_{r2})$, to vary in the set \{0.5, 1.0, 1.5, 2.0, 2.5, 3.0\} (DPS scatterers) or \{-0.5, -1.0, -1.5, -2.0, -2.5, -3.0\} (DNG scatterers), whereas in the second part we consider a particular value of $\Re(\varepsilon_{r2}) < 0$ and $\Re(\mu_{r2}) < 0$ but for different meshes and different losses of the materials involved.

The results obtained in the first part of our analysis are shown in Figure 1 for DPS scatterers and in Figure 2 for DNG ones. These figures show that the mean values of the absolute or relative errors along the three coordinate axes can be much bigger for DNG scatterers than for DPS ones. For example, the average relative error is in the range [0.12, 0.56] for spheres made up of DNG metamaterials and belongs to [0.025, 0.067] when these spheres are made up of DPS media, with an increase of the average value of almost one order of magnitude. Moreover, for DNG scatterers the accuracy is not related to the “density” of the scatterer which, on the contrary, is a well known effect for DPS scattering objects (as shown in Figure 1).

Figure 2 shows that significant information could be provided by considering cases characterized by $\Re(\varepsilon_{r2})$ and $\Re(\mu r2)$ around -1.0. If $\Re(\varepsilon_{r2}) = \Re(\mu r2) = -1.0$ the media in the investigation domain are said to be conjugate [21]. Even if the errors achieved in this case are not the highest ones among all the DNG scatterers considered, this example is seen to suffer from a very significant deterioration in the accuracy of the finite element simulator with respect to any tested DPS scatterer ($\Im(rmx) = 0.303$, $\Im(rmy) = 0.253$, $\Im(rmz) = 0.282$) and it could be chosen as a test case for a more detailed analysis on this phenomenon.

Thus, in the following, we will study scatterers made up of DNG metamaterials with $\Re(\varepsilon_{r2}) = \Re(\mu_{r2}) = -1.0$, with different values of the imaginary parts and with different meshes. The performances of finite element simulators when DPS scatterers are involved are again considered just as terms of comparison.

Some of the obtained results have been reported in Figure 3, where the average absolute and relative errors along the three lines of points described above are shown as a function of the discretization parameter $n$. It can be noted that when DNG scatterers are involved both figures of merit are always worse than the corresponding quantity calculated with DPS scatterers, even when the DPS scatterer is much denser (with $\Re(\varepsilon_{r2}) = 3.0$). The deterioration with respect to the much denser DPS scatterer can be small in the case the imaginary parts of $\varepsilon_{r1}$ and $\mu_{r2}$ are equal to -0.9 but becomes impressive when the same imaginary parts are equal to -0.3 or -0.1. For example, the relative errors obtained when the DNG scatterer is involved, $\Im(\varepsilon_{r1}) = \Im(\mu_{r2}) = -0.3$ and $n = 51$ (the mesh has nearly 100 elements per minimum wavelength) are comparable to those obtained with the strongest DPS scatterer among those considered ($\Re(\varepsilon_{r2}) = 3.0$) with $n = 11$ (the mesh has slightly more than 10 elements per minimum wavelength). Moreover, depending on the chosen figure of merit and line of points, when a mesh with $n = 51$ is considered the accuracy in the presence of DNG scatterer is reduced by a factor in the range [1.1, 2.0] when $\Im(\varepsilon_{r1}) = \Im(\mu_{r2}) = -0.9$, in the range [2.0, 5.3] when $\Im(\varepsilon_{r1}) = \Im(\mu_{r2}) = -0.3$ and in the range [3.3, 15.2] when $\Im(\varepsilon_{r1}) = \Im(\mu_{r2}) = -0.1$ with respect to the case showing the highest errors among DPS scatterers, that is the case in which the DPS medium in the sphere is lossless and has $\Re(\varepsilon_{r2}) = 3.0$. This factor is even bigger if one considers, as terms of comparison, other DPS scatterers. Finally, it could be interesting to note that the accuracy reduction pointed out in [15], [16] for 1D and 2D problems was much
Figure 1: Behaviour of (a) $E_{amx}$, (b) $E_{amy}$, (c) $E_{amz}$, (d) $E_{ermx}$, (e) $E_{ermx}$, (f) $E_{ermz}$, for different DPS scatterers. In particular, $\text{Re}(\varepsilon_{r1}) = 1.0$, $\text{Re}(\mu_{r1}) = 1.0$, $\text{Im}(\varepsilon_{r1}) = -0.3$, $\text{Im}(\mu_{r1}) = 0.0$, $\text{Im}(\varepsilon_{r2}) = 0.0$, $\text{Im}(\mu_{r2}) = -0.3$ and $n = 31$, whereas $\text{Re}(\varepsilon_{r2})$ and $\text{Re}(\mu_{r2})$ vary in the set \{0.5, 1.0, 1.5, 2.0, 2.5, 3.0\}. 
“On the performances of electromagnetic finite element...”

Figure 2: Behaviour of (a) $E_{amx}$, (b) $E_{amy}$, (c) $E_{amz}$, (d) $E_{ermx}$, (e) $E_{erm}$, (f) $E_{ermz}$, for different DNG scatterers. In particular, $\text{Re}(\varepsilon_{r1}) = 1.0$, $\text{Re}(\mu_{r1}) = 1.0$, $\text{Im}(\varepsilon_{r1}) = -0.3$, $\text{Im}(\mu_{r1}) = 0.0$, $\text{Im}(\varepsilon_{r2}) = 0.0$, $\text{Im}(\mu_{r2}) = -0.3$ and $n = 31$, whereas $\text{Re}(\varepsilon_{r2})$ and $\text{Re}(\mu_{r2})$ vary in the set $\{-0.5, -1.0, -1.5, -2.0, -2.5, -3.0\}$. 
Figure 3: Behaviour of $E_{\text{rms}}$ and $E_{\text{rms}}$, for different discretizations, different values of $\varepsilon_2$ and $\mu_2$ and different values of the imaginary part of $\varepsilon_1$. The real parts of $\varepsilon_{r1}$ and $\mu_{r1}$ are equal to 1.0 in all cases.
smaller than that deduced above.

So far we have considered the performances in terms of mean values of the errors. However, as already pointed out, also the local accuracy of the approximation is of interest. For this reason in Figure 4 we report the behaviour of the $x$ component of the electric field for different DNG scatterers. For comparison in Figure 5 we report the behaviour of the same component in the case of a dense DPS scatterer ($\text{Re}(\varepsilon_{r1}) = \text{Re}(\mu_{r1}) = 3.0$).

Parts (a) and (b) of Figure 4 show the electric field behaviour when the imaginary parts of $\varepsilon_{r1}$ and $\mu_{r1}$ are equal to $-0.9$. Parts (c) and (d) of Figure 4 correspond to $\text{Im}(\varepsilon_{r1}) = \text{Im}(\mu_{r1}) = -0.3$. Finally, parts (e) and (f) of the same Figure report the electric field behaviour when the imaginary parts of $\varepsilon_{r1}$ and $\mu_{r1}$ are equal to $-0.1$.

Figures 4 and 5 confirm that the quality of the approximation is worse and sometimes terribly worse when DNG scatterers are involved. Moreover they show that the accuracy of the approximation is highly affected by the magnitude of the loss tangents when DNG scatterers are involved. This effect is particularly evident along the $x$ axis (parts (a), (c) and (e) of Figure 4). Finally, it is worthy to note that the error is big especially at the interface between the DPS and DNG media. This phenomenon could be related to the comment reported in [17] concerning the fact that the field at the air/metamaterial interface is very strong and that “an extremely dense mesh is needed to obtain accurate results”.

In summary, our results show that it could be sensible to use, when DNG metamaterials are involved, much finer meshes than those usually exploited when “standard” DPS media are considered. Even if the effect is evident also for high losses (when $\text{Im}(\varepsilon_{r1}) = \text{Im}(\mu_{r1}) = -0.9$), this aspect is particularly critical when smaller values of losses are considered. Moreover, it is important to note that the amplitude of the error on $E_x$ around the DPS/DNG interface decreases very slowly as $n$ increases.

5  Performances of the BCGM iterative solver

As stated in the Introduction, one of the purposes of this paper is to give some indications on the behaviour of finite-elements-based simulators in terms of their speed of convergence in the presence of DNG and DPS materials.

It is well known from the literature that iterative methods are efficiently used to solve large matrix equations. The most popular iterative solver used for the solution of linear systems resulting from FEM discretizations is that based on the biconjugate gradient method (BCGM) [12, 22].

The BCGM has been extensively used in [15] and [16] for 1D and 2D analyses. It was proved that the BCGM can be much slower when handling DNG materials than with standard DPS media. In order to provide more general indications on the performances of this kind of solver, here we show the results obtained by using BCGM on 3D problems involving both DPS and DNG materials.

The stopping criteria we have always adopted, which defines when the iterative solution has reached convergence, is “criterion 2” of [22] (p. 60). For the reader convenience, if the algebraic linear system to be solved is $Ax = b$, being $A$ the finite element matrix [12], $b$ the known term due to the electromagnetic sources or boundary conditions [12] and $x$ the algebraic vector of finite element degrees of freedom [12], we initially calculate the euclidean norm $\|b\|$ and do not stop the iterative solver until the approximate solution $x_i$, at iteration $i$ satisfies $\|Ax_i - b\| < \delta|b|$, with $\delta$ always equal to $10^{-10}$.

In order to make fair comparisons between the CPU times, all the simulations were performed on an AMD Athlon 64 Processor 3500+ working at a frequency of 2202.924 MHz with 512 KB of cache memory and 4 GB of RAM memory.

In Figure 6 we report the CPU times (in seconds) necessary to compute the finite element approximations for different scatterers and different discretizations, even though not all results obtained are shown. In particular, for the cases reported in Figure 6, the external medium was always characterized by $\text{Re}(\varepsilon_{r1}) = \text{Re}(\mu_{r1}) = 1.0$ and we considered DNG scatterers with $\text{Re}(\varepsilon_{r2}) = \text{Re}(\mu_{r2}) = -1.0$, $\text{Im}(\varepsilon_{r2}) = \text{Im}(\mu_{r2}) = 0.0$ and $\text{Im}(\varepsilon_{r1}) = \text{Im}(\mu_{r1}) = -0.1, -0.3, -0.9$. We con-
Figure 4: Amplitude of the $x$ component of the electric field along the line of points $(x, y = 0.0003, z = 0.003)$ (a), (c), (e) and $(x = 0.003, y = 0.003, z)$ (b), (d), (f). Different discretizations are considered. A DNG scatterer is involved having the real parts of $\varepsilon_r^2$ and $\mu_r^2$ equal to $-0.0.0$. The real parts of $\varepsilon_r^1$ and $\mu_r^1$ are equal to 1.0, the imaginary parts of $\varepsilon_r^2$ and $\mu_r^1$ are trivial and the imaginary parts of $\varepsilon_r^1$ and $\mu_r^2$ are equal to $-0.9$ (a), (b), $-0.3$ (c), (d), $-0.1$ (e), (f).
Figure 5: Amplitude of the $x$ component of the electric field along the line of points $(x, y = 0.0003, z = 0.003)$ (a) and $(x = 0.003, y = 0.003, z)$ (b). Different discretizations are considered. A DPS scatterer is involved having the real parts of $\varepsilon_{r,2}$ and $\mu_{r,2}$ equal to 3.0. The real parts of $\varepsilon_{r,1}$ and $\mu_{r,1}$ are equal to 1.0 and all the imaginary parts of $\varepsilon_{r,1}, \mu_{r,1}, \varepsilon_{r,2}$ and $\mu_{r,2}$ are trivial.

Considered also DPS scatterers with $Re(\varepsilon_{r,2}) = Re(\mu_{r,2}) = 1.5, 2.0, 3.0, Im(\varepsilon_{r,2}) = Im(\mu_{r,1}) = 0.0$ and $Im(\varepsilon_{r,1}) = Im(\mu_{r,2}) = 0.0, -0.1, -0.3, -0.9$.

It is easy to see that the CPU time needed for the simulation of a DNG scatterer is heavily dependent on the losses inserted in the involved materials. This effect is not significant when DPS scatterers are considered. It is also important to note that, in the presence of DNG materials, the CPU time needed to solve the algebraic systems quickly rises with the number of unknowns, reaching values that are dramatically high on normal PCs even when the number of unknowns is not prohibitive to handle if compared with analogous problems involving only DPS scatterers. For example, for $n = 51$ the finite element solution can be calculated in less than 52 minutes (3117.2 s) for a DPS scatterer but it could take 8.9 days (768892.2 s) to perform the same calculation for a DNG scatterer! Thus the increase of CPU time can be equal to $\simeq 250$, which is much bigger than the value ($\simeq 20$) obtained in [15] for 1D scatterers and of the same order of magnitude of the value obtained (with a much lower level of losses, however) for 2D scatterers [16].

The results outlined above prove that the speed of convergence of finite element simulators can be impressively worsened by the presence, at the same time, of DPS and DNG materials with respect to cases involving only DPS media. A possible explanation for the observed effects is that the condition number of the finite element matrix could be much worse than usual when DNG media and DPS materials are involved. As a matter of fact, the entry on the main diagonal of the finite element matrix corresponding to the edge element function $v_i$ associated to edge number $i$ placed on the interfaces between DPS and DNG materials (and not on $\Gamma$) is given by

$$\int_{\Omega} \mu^{-1}(\nabla \times v_i) \cdot (\nabla \times v_i^*) \, dV - \omega^2 \int_{\Omega} \varepsilon v_i \cdot v_i^* \, dV$$

(8)

Suppose that the support of $v_i$ is symmetric with respect to the interface. If on both sides of the interface we have DPS or DNG media the single integrals $\int_{\Omega} \mu^{-1}(\nabla \times v_i) \cdot (\nabla \times v_i^*) \, dV$ and $\int_{\Omega} \varepsilon v_i \cdot v_i^* \, dV$ cannot be zero but when $\varepsilon$ and $\mu$ are real, of opposite sign on the two sides of the interface and with the same absolute value both the above integrals are equal to zero. The losses help to avoid this situations since the sign of the imaginary parts of $\varepsilon$ and $\mu$ is always the same independently of the fact that the media is DPS or DNG. This explains also why with a uniform mesh the performances of the BCGM are more critical when $Re(\varepsilon_{r,2}) = Re(\mu_{r,2}) \simeq -1.0$ than when other cases are considered.

Finally, it is worthy to note that we have always avoided using preconditioning. This is due mainly to the fact that too many such techniques exist and we have not attempted to obtain general indications on this particular point. However, some comments could be made, starting from
Figure 6: CPU time required by the finite element simulator to compute the solutions of the problems of interest.
the considerations given above, which confirm that in some cases the use of preconditioners can deteriorate the performances of the simulators in presence of DNG media. Actually, all preconditioners which suppose the entries on the main diagonal of the finite element matrix different from zero can worsen the problem. The simplest preconditioning technique known as “point Jacobi preconditioning” [22] builds a diagonal matrix with entries given by $1/a_{ii}$, where $a_{ii}$ denotes the entry on the diagonal of the matrix $A$ on the $i$-th row. This preconditioner can give very bad performances when some diagonal entries can be very small as indicated above. We have verified that the biconjugate gradient method solver with such a preconditioner gives very worse performances in terms of CPU time when the losses are small.

6 Conclusions

When finite element simulators are exploited, it is common practice to generate the triangulation in such a way that the elements have a maximum linear dimension equal to a fraction of the wavelength. The outcome of this paper is that, in order to obtain a given accuracy when DNG scatterers are considered, it can be sensible to use meshes much finer than those generated according to the previous common feeling, at least in regions containing and slightly surrounding the scatterer. In particular, this considerations are of importance when the losses of the DNG metamaterials involved are small.

In this work, moreover, it is pointed out that DNG metamaterials can have an impressive impact on the performances of the most popular iterative solver, possibly making finite element simulators much slower than usual.

The results pointed out in this work confirm the fact, shown in [15] and [16] for the first time to the best of authors’ knowledge, that the presence of DNG metamaterials can affect the performances of 3D finite element simulators. However, from a quantitative point of view the results obtained in this work for more realistic scattering problems are much worse than those previously deduced.

Future efforts will be dedicated to the possible countermeasures to be adopted against the effects pointed out in this work.

References


[21] A. Alú and N. Engheta, “Guided modes in a waveguide filled with a pair of single-negative (SNG), double-negative (DNG) and/or double positive (DPS) layers,” IEEE *Transactions on Microwave Theory and Techniques*, vol. 52, pp. 199–210, January 2004.