Unique solvability for electromagnetic boundary value problems in the presence of partly lossy inhomogeneous anisotropic media and mixed boundary conditions

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Abstract
A result on the existence and uniqueness of the solutions of electromagnetic boundary value problems is presented. It is a generalization of the results already available in the open literature which holds true in all cases of practical interest. As a matter of fact, it holds true even in the presence of general inhomogeneous anisotropic media, surfaces of discontinuity, topologically complicated domains and mixed boundary conditions.

1 Introduction
During the second half of the last century many authors have addressed the question of existence and/or uniqueness of the solution of electromagnetic boundary value problems.

On the one hand, among mathematicians, Müller [1], Leis [2], Weck and Witsch [3], Bossavit [4, 5], Cessenat [6] and Alonso and Valli [7] have stated elegant results concerning both existence and uniqueness of the solutions of some boundary value problems of interest in electromagnetics. Unfortunately, for many problems of practical interest, especially at microwave frequencies, all these elegant results cannot be exploited. As a matter of fact, Müller [1] and Bossavit [4, 5] considered piecewise homogeneous isotropic materials; Leis [2] and Cessenat [6] dealt with anisotropic inhomogeneous media, but only in the cases of everywhere lossy or everywhere lossless materials; Alonso and Valli [7] did not allow for jump discontinuities in lossless materials. Weck and Witsch [3] overcame much of the above limitations and provided the most general result in this community but even this contribution considered neither mixed boundary conditions nor general complex dielectric parameters. Unfortunately, in many microwave problems, both types of boundary conditions are necessarily involved. This is the case, for example, when the magnetic field is known on the apertures of a waveguide junction, being the tangential electric field forced to zero on the rest of the boundary, or when, by exploiting symmetries and aiming at a reduction of the computational cost of the calculation of an approximate solution by a numerical method, the problem is posed just on a part of the domain over which the electromagnetic phenomena to model take place. Moreover, complex dielectric parameters are necessary to model ionized gases subject to steady magnetic fields, which are of importance in plasma physics and are involved in the study of the propagation of electromagnetic waves through the ionosphere [8], and to model ferrite materials, which are exploited in a large number of microwave devices [9]. Therefore, even the elegant and interesting result by Weck and Witsch [3] is unable to establish the existence and uniqueness of the solutions of many practically important electromagnetic boundary value problems.

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On the other hand, in the electromagnetic engineering community, Harrington [10], Collin [11], Balanis [12], Chu and Liang [13], Caorsi and Raffetto [14–17], Lei et al. [18], and others have stated results which hold true, in some cases, in more general situations but all these contributions suffer from two major drawbacks: the results are just in terms of uniqueness and cannot be generalized to considerations on the existence of the solution and, in an implicit or explicit way, some regularity of the solutions is a priori assumed in order for the proofs of uniqueness to hold true.

Thus our target is to provide a result on the existence and uniqueness of the solutions of electromagnetic boundary value problems which, by overcoming all drawbacks of the cited contributions, holds true in (almost) all cases of practical interest. In particular, the domains where the electromagnetic boundary value problems will be posed can be topologically complicated, the boundary conditions can be of mixed type and, as far as material properties are considered, we will manage general anisotropic inhomogeneous complex dielectric parameters.

The proof of this more general result extends the one in [7] which is valid for Dirichlet boundary conditions and material properties with $C^2$ regular coefficients. A crucial point in [7] is the validity of a compactness result for vector fields with divergence and curl in $L^2$ that extends the classical results of Weber [19] and Picard [20]. In this work, we extend the compactness result to the case of mixed boundary conditions using the set of results obtained by Fernandes and Gilardi in [21], and in order to manage the discontinuity of the coefficients, we use the technique introduced by Hazard and Lenoir in [22].

The paper is organized as follows: in the next section we specify the assumptions we require on the domain, on the dielectric parameters and on the data of the problem. In Section 3 we define the problem and state the main result of this work. Moreover we transform the original problem into an equivalent one with which we will work in the sequel. In Section 4 we introduce the weak formulation of the problem stated in Section 3 and, as in [7], we prove its unique solvability in two steps: first, as a consequence of a suitable compactness result, proven in Section 5, we show the problem and state the main result of this work. Moreover we transform the original problem into an equivalent one with which we will work in the sequel. In Section 4 we introduce the weak formulation of the problem stated in Section 3 and, as in [7], we prove its unique solvability in two steps: first, as a consequence of a suitable compactness result, proven in Section 5, we show that the Fredholm alternative holds for such a problem; second, we show that the kernels of the considered operator and its adjoint are trivial.

2 Assumptions and notations

Let $\Omega \subset \mathbb{R}^3$ be the open, bounded and connected Lipschitz domain where the electromagnetic boundary value problem of interest will be posed. Let $\Gamma = \partial \Omega$ be its boundary and suppose it splits into two disjoint open subsets $\Gamma_\tau$ and $\Gamma_\nu$. $\Gamma_\tau$ and $\Gamma_\nu$ are compact Lipschitz submanifolds of $\Gamma$ satisfying $\Gamma_\tau \cup \Gamma_\nu = \Gamma$. The limit cases $\Gamma_\tau = \emptyset$ and $\Gamma_\nu = \Gamma$ or $\Gamma_\nu = \emptyset$ and $\Gamma_\tau = \Gamma$ are allowed. The outward unit vector normal to $\Gamma$ will be denoted by $n$.

Let us assume, moreover, that a set of cuts $\Sigma_j$, $j = 1, \ldots, N$, satisfying the following conditions [21] can be introduced. Each cut $\Sigma_j$ is the interior of a compact and connected two-dimensional Lipschitz manifold $\Sigma_j$ such that $\Sigma_j \subset \Omega$ and $\partial \Sigma_j \subset \Gamma$. Moreover, the manifolds $\Sigma_j$ are mutually disjoint and for any $j$ there exists a connected open set $\Omega_j \supset \Sigma_j$ such that $\Omega \cap \Omega_j \setminus \Sigma_j$ has exactly two connected components and each of them is a Lipschitz domain whose boundary contains $\Sigma_j$. This implies in particular that the cuts are globally two-sided and the set $\bar{\Omega} = \Omega \setminus \Sigma$, with $\Sigma = \bigcup_{j=1}^N \Sigma_j$, behaves locally as a Lipschitz domain, provided that its boundary is thought as made by $\Gamma \setminus \partial \Sigma$ plus two distinct copies of $\Sigma$. As a last hypothesis about the cuts, all of them are required in order that in $\Omega$ every curl-free vector field has a global scalar potential. Finally, we assume that both $\Gamma_\tau \setminus \partial \Sigma$ and $\Gamma_\nu \setminus \partial \Sigma$ have a finite number of connected components and that the closure of each of them is a Lipschitz two-dimensional submanifold of $\Gamma$.

In order to model different inhomogeneous anisotropic materials, let $\eta^*$ denote the conjugate transpose of a generic 3-by-3 complex matrix $\eta$. Moreover, let $\eta_\tau$ and $\eta_\nu$ be the hermitian symmetric matrices such that $\eta = \eta_\nu - i \eta_\nu s = \frac{\eta + \eta^*}{2} - i \frac{\eta - \eta^*}{2}$. We will say that a 3-by-3 hermitian symmetric matrix-valued complex function $\eta(x)$ is uniformly positive definite in $\Omega$ if there exists a positive constant $\eta_0$ such that $\eta^* (x) \eta \geq \eta_0 |a|^2$ a.e. in $\Omega$, $\forall a \in \mathbb{C}^3$. Finally, let $A_\nu$, $A_\tau$ and $\Omega_k$, $k = 1, \ldots, L$, be open Lipschitz subdomains of $\Omega$ satisfying $\bar{\Omega} = \overline{A_\nu} \cup \overline{A_\tau} \cup \overline{\Omega}_1 \cup \ldots \cup \overline{\Omega}_L$ and such that the interior of $\overline{A_\nu} \cup \overline{A_\tau}$ is non empty. With these notations, let $\varepsilon$ and $\mu$ be two
3-by-3 matrix-valued complex functions representing the equivalent dielectric permittivity and the magnetic permeability, respectively, and assume that:

**H1.** all entries of \( \varepsilon \) and \( \mu \) are in \( L^\infty(\Omega) \)

**H2.** \( \varepsilon_s \) and \( \mu_s \) are uniformly positive definite in \( \Omega \)

**H3.** \( \varepsilon_{ss} \) is uniformly positive definite in \( A_e \)

**H4.** \( \varepsilon_{ss} = 0 \) in \( \Omega \setminus A_e \)

**H5.** \( \mu_{ss} \) is uniformly positive definite in \( A_m \)

**H6.** \( \mu_{ss} = 0 \) in \( \Omega \setminus A_m \)

**H7.** \( \varepsilon_{mn|_{\Gamma_k}} = \mathcal{E}_{mn|_{\Gamma_k}}^k \) and \( \mu_{mn|_{\Gamma_k}} = \mathcal{M}_{mn|_{\Gamma_k}}^k \) with \( \mathcal{E}_{mn}^k, \mathcal{M}_{mn}^k \in C^2(\mathbb{R}^3) \) for each \( k = 1, \ldots, L \), \( 1 \leq m, n \leq 3 \).

Besides the trivial interpretation of condition (H1), from a physical point of view, these hypotheses concerning material properties mean that the anisotropic medium may present surfaces of discontinuity (H1 and H7) and that it is electrically lossy in \( A_e \) (H3), magnetically lossy in \( A_m \) (H5) and passive and lossless in \( \Omega \setminus (A_e \cup A_m) \) (H4 and H6). The positiveness of \( \varepsilon_s \) and \( \mu_s \) is explicitly assumed (H2) as usual [6]. This assumption is technically necessary for our future developments and makes it possible to avoid problems supporting unusual electromagnetic phenomena [23, 24], even though this precludes the possibility to deal with some models of interest (a model of a ionized gas [25], for example). The local \( C^2 \) regularity (H7) is again technically necessary but does not seem a real restriction since the dielectric parameters are just piecewise but not globally \( C^2 \).

Finally, note that the cases \( \Omega = A_e \) and \( A_e \cup A_m = \emptyset \) have already been considered by Leis ([2]).

In the following sections the angular frequency will be denoted by \( \alpha \). We will always assume that \( \alpha \neq 0 \).

Finally, let \( H(\text{rot}; \Omega) \) be the space of complex vector fields \( v \) belonging to \( (L^2(\Omega))^3 \) such that also \( \text{rot}v \in (L^2(\Omega))^3 \). If \( \Lambda \) is a Lipschitz submanifold of \( \Gamma \) let us denote by \( \chi_\Lambda \) the following space (its intrinsic characterization when \( \Lambda = \Gamma \) is provided in [6] when \( \Omega \) is a regular domain and in [26] when \( \Omega \) is a Lipschitz polyhedra):

\[
\chi_\Lambda := \{ (v \times \mathbf{n})|_{\Lambda} \mid v \in H(\text{rot}; \Omega) \} \subset (H^{-1/2}(\Lambda))^3.
\]

Other spaces will be introduced later on when necessary.

## 3 Statement of the problem

With the notations introduced in Section 2, let us now consider the following electromagnetic boundary value problem:

\[
\begin{align*}
\mathbf{E} & \in L^2(\Omega) \\
\mathbf{H} & \in L^2(\Omega) \\
\mathbf{E} \times \mathbf{n} & = f_D \quad \text{on } \Gamma_D \\
\mathbf{H} \times \mathbf{n} & = f_N \quad \text{on } \Gamma_N \\
i \alpha \varepsilon & \mathbf{E} - \text{rot} \mathbf{H} = \mathbf{F} \quad \text{in } \Omega \\
i \alpha \mu & \mathbf{H} + \text{rot} \mathbf{E} = \mathbf{G} \quad \text{in } \Omega
\end{align*}
\]

(1)

where all types of possible sources have been considered, since all of them may be important in equivalent formulations of electromagnetic problems [12].

Due to the generality of our hypotheses, no linear electromagnetic boundary value problem of practical interest is excluded, with the only exception of those involving chiral or bianisotropic materials [27]. In particular, let us point out that even problems posed in topologically complicated domains and involving mixed boundary conditions and general anisotropic dielectric media can be dealt with.

The aim of this work is to prove the following result:
Theorem 1. Under the hypotheses reported in Section 2, for any $F$ and $G$ in $(L^2(\Omega))^3$ and for any $f_D \in \chi_{\Gamma_D}$ and $f_N \in \chi_{\Gamma_N}$, problem (1) has a unique solution.

Without loss of generality we can assume $f_D = f_N = 0$. In fact, if $f_D$ and $f_N$ are in $\chi_{\Gamma_D}$ and $\chi_{\Gamma_N}$, respectively, from the definition of the space $\chi_{\Lambda}$ there exist $Rf_D$ and $Rf_N \in H(\text{rot}; \Omega)$ such that $(Rf_D \times n)_{|\Gamma_D} = f_D$ and $(Rf_N \times n)_{|\Gamma_N} = f_N$. Setting $\tilde{E} = E - Rf_D$ and $\tilde{H} = H - Rf_N$ we are left with the solvability of the following problem:

\[
\begin{cases}
\ i \alpha \tilde{E} - \text{rot} \tilde{H} = \tilde{F} & \text{in } \Omega \\
\ i \alpha \mu \tilde{H} + \text{rot} \tilde{E} = \tilde{G} & \text{in } \Omega \\
\tilde{E} \times n = 0 & \text{on } \Gamma_\tau \\
\tilde{H} \times n = 0 & \text{on } \Gamma_\nu
\end{cases}
\]

where $\tilde{F} = F - (i \alpha Rf_D - \text{rot} Rf_N) \in (L^2(\Omega))^3$ and $\tilde{G} = G - (i \alpha \mu Rf_N + \text{rot} Rf_D) \in (L^2(\Omega))^3$.

Under assumption (H2) it is possible to eliminate one of the fields $\tilde{E}$ or $\tilde{H}$. Eliminating for instance the unknown $\tilde{H}$ we have to deal with the equivalent boundary value problem

\[
\begin{cases}
\ i \alpha \tilde{E} - \text{rot} \tilde{E} = \tilde{F} & \text{in } \Omega \\
\tilde{E} \times n = 0 & \text{on } \Gamma_\tau \\
\mu^{-1} \text{rot} \tilde{E} \times n = \mu^{-1} \tilde{G} \times n & \text{on } \Gamma_\nu
\end{cases}
\]

(2)

If $\mu$ satisfies assumptions (H1), (H2), (H5) and (H6), $\mu^{-1}$ is a matrix-valued complex function with entries in $L^\infty(\Omega)$. Moreover, setting $\nu_s := \frac{\mu^{-1} + (\mu^{-1})^*}{2}$ and $\nu_{ss} := \frac{\mu^{-1} - (\mu^{-1})^*}{2i}$ we have $\nu^{-1} = \nu_s + i\nu_{ss}$; $\nu_s$ is uniformly positive definite in $\Omega$ and $\nu_{ss}$ is uniformly positive definite in $\operatorname{Im}_m$ and equal zero in $\Omega \setminus \overline{\operatorname{Im}_m}$.

We note that in problem (2) it is possible to assume $\tilde{G} = 0$ without loss of generality. In fact, if assumption (H2) is satisfied, there exists a unique solution $W$ of the following problem

\[
\begin{cases}
\ i \alpha \tilde{E} - \text{rot} W = \tilde{F} & \text{in } \Omega \\
W \times n = 0 & \text{on } \Gamma_\tau \\
\mu^{-1} \text{rot} W \times n = \mu^{-1} \tilde{G} \times n & \text{on } \Gamma_\nu
\end{cases}
\]

Then setting $U = \tilde{E} - W$, the unique solvability of problem (2) is equivalent to the unique solvability of the following problem

\[
\begin{cases}
\ i \alpha \tilde{E} - \text{rot} U = \tilde{F} + (1 + \alpha^2 \varepsilon) \tilde{W} & \text{in } \Omega \\
U \times n = 0 & \text{on } \Gamma_\tau \\
\mu^{-1} \text{rot} U \times n = 0 & \text{on } \Gamma_\nu
\end{cases}
\]

(3)

with $M = \alpha \tilde{F} + (1 + \alpha^2 \varepsilon) \tilde{W} \in (L^2(\Omega))^3$.

In Section 4 we will prove the following result:

Theorem 2. Let the hypotheses reported in Section 2 be satisfied; if $M \in (L^2(\Omega))^3$ is such that $\text{div} M = 0$ in $\Omega$ and $M \cdot n = 0$ on $\Gamma_\nu$ then problem (3) has a unique solution.

We note that the assumption $\text{div} M = 0$ in $\Omega$ and $M \cdot n = 0$ on $\Gamma_\nu$ is not restrictive. In fact, we can proceed as in [7]: the boundary value problem

\[
\begin{cases}
\ i \alpha \text{rot} \varphi = \text{div} M & \text{on } \Omega \\
\varphi = 0 & \text{on } \Gamma_\tau \\
\text{rot} \varphi \cdot n = M \cdot n & \text{on } \Gamma_\nu
\end{cases}
\]

has a unique solution $\varphi \in H^1(\Omega)$ (if $\Gamma_\tau = \emptyset$, in order to have a unique solution, we ask $\int_{\Gamma_\nu} \varphi = 0$).

Setting $\tilde{M} := M - \varepsilon \text{rot} \varphi$, which clearly satisfies $\text{div} \tilde{M} = 0$ in $\Omega$ and $\tilde{M} \cdot n = 0$ on $\Gamma_\nu$, we are left
with the solvability of problem

\[
\begin{aligned}
\begin{cases}
\text{rot}(\mu^{-1}\text{rot} \hat{U}) - \alpha^2 \hat{U} = \tilde{M} & \text{in } \Omega \\
\hat{U} \times n = 0 & \text{on } \Gamma_{\tau} \\
\mu^{-1}\text{rot} \hat{U} \times n = 0 & \text{on } \Gamma_{\nu}.
\end{cases}
\end{aligned}
\]

Then \( U = \hat{U} - \alpha^{-2} \nabla \varphi \).

### 4 Existence and uniqueness

We start introducing some functional spaces that we will use in the sequel. Some of them depend on a weight \( \omega \) that in this work will be \( \varepsilon \) or the \( 3 \times 3 \) identity matrix \( I \).

\[
\begin{align*}
H(\omega, \text{div}; \Omega) := \{ v \in (L^2(\Omega))^3 \mid \text{div}(\omega v) \in L^2(\Omega) \} \\
H^0(\omega, \text{div}; \Omega) := \{ v \in H(\omega, \text{div}; \Omega) \mid \text{div}(\omega v) = 0 \} \\
H_0,\Gamma_\tau(\omega, \text{div}; \Omega) := \{ v \in H(\omega, \text{div}; \Omega) \mid \langle \omega v \rangle \cdot n_{\Gamma_\tau} = 0 \} \\
H^0_0,\Gamma_\tau(\omega, \text{div}; \Omega) := H^0(\omega, \text{div}; \Omega) \cap H_0,\Gamma_\tau(\omega, \text{div}; \Omega) \\
H^0(\text{rot}; \Omega) := \{ v \in H(\text{rot}; \Omega) \mid \text{rot} v = 0 \} \\
H_0,\Gamma_\nu(\text{rot}; \Omega) := \{ v \in H(\text{rot}; \Omega) \mid v \times n_{\Gamma_\nu} = 0 \} \\
H^0_0,\Gamma_\nu(\text{rot}; \Omega) := H^0(\text{rot}; \Omega) \cap H_0,\Gamma_\nu(\text{rot}; \Omega)
\end{align*}
\]

\((s = \tau, \nu)\). If \( \omega = I \)

\[
\begin{align*}
H(\omega, \text{div}; \Omega) := H(\text{div}; \Omega) \\
H^0(\omega, \text{div}; \Omega) := H^0(\text{div}; \Omega) \\
H_0,\Gamma_\tau(\omega, \text{div}; \Omega) := H_0,\Gamma_\tau(\text{div}; \Omega) \\
H^0_0,\Gamma_\tau(\omega, \text{div}; \Omega) := H_0,\Gamma_\tau(\text{div}; \Omega).
\end{align*}
\]

We introduce in \( H(\text{rot}; \Omega) \) the bilinear form

\[
a(w, v) := (\mu^{-1}\text{rot} w, \text{rot} v) - \alpha^2(\varepsilon w, v)
\]

where \((\cdot, \cdot)\) is the scalar product in \((L^2(\Omega))^3\) for complex valued vector functions.

The weak formulation of problem (3) reads:

\[
\begin{aligned}
\text{Find } U \in H_0,\Gamma_\tau(\text{rot}; \Omega) : \\
\begin{cases}
\text{subject to } a(U, v) = (M, v) \quad \forall v \in H_0,\Gamma_\tau(\text{rot}; \Omega).
\end{cases}
\end{aligned}
\]

(4)

In order to prove Theorem 2 we start with the following result:

**Theorem 3.** Let (H1), (H2) and the hypotheses on \( \Omega \) and \( \Gamma \) reported in Section 2 be satisfied. If \( M \in H_0^0,\Gamma_\tau(\text{div}; \Omega) \) then the Fredholm alternative holds for the boundary value problem (4) and the necessary and sufficient solution condition is \((M, \Phi) = 0\) for all solutions \( \Phi \in H_0,\Gamma_\tau(\text{rot}; \Omega) \) of the adjoint problem

\[
((\mu^{-1})^*\text{rot} \Phi, \text{rot} v) - \alpha^2(\varepsilon^* \Phi, v) = 0 \quad \forall v \in H_0,\Gamma_\tau(\text{rot}; \Omega).
\]

(5)

**Proof.** The proof is analogous to the one of Theorem 2.4 in [7]. The main point is to show that the immersion \( H_0,\Gamma_\tau(\text{rot}; \Omega) \cap H_0,\Gamma_\nu(\varepsilon, \text{div}; \Omega) \hookrightarrow (L^2(\Omega))^3 \) is compact. This technical result will be proved in the following section.

Since \( \varepsilon_s \) and \( \nu_s \) are uniformly positive definite in \( \Omega \), the auxiliary bilinear form

\[
\tilde{a}(w, v) := (\mu^{-1}\text{rot} w, \text{rot} v) + (\varepsilon w, v)
\]

\[
= a(w, v) + (1 + \alpha^2)(\varepsilon w, v)
\]

is coercive in \( H(\text{rot}; \Omega) \). Therefore for each \( f \in (L^2(\Omega))^3 \) there exists a unique \( Sf \in H_0,\Gamma_\tau(\text{rot}; \Omega) \) such that

\[
\tilde{a}(Sf, v) = (f, v) \quad \forall v \in H_0,\Gamma_\tau(\text{rot}; \Omega)
\]
We report it here for the sake of completeness.

We start by noticing that $H$ space with respect to the scalar product of $H$ follows that $S$ is the adjoint operator of $\varepsilon$ denoting by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, and using that $\nabla \psi \in H_{0, \Gamma, (\text{rot}; \Omega)}$, we have

$$\langle \varepsilon S f, \psi \rangle = \langle \text{div}(\varepsilon S f), \psi \rangle + \langle \varepsilon S f, \nabla \psi \rangle = \langle \text{div}(\varepsilon S f), \psi \rangle + \langle f, \nabla \psi \rangle - (\mu^{-1} \text{rot} S f, \text{rot} \nabla \psi) = 0$$

if $f \in H_{0, \Gamma, (\text{div}; \Omega)}^0$.

Hence, if we define the operator $S_\varepsilon : (L^2(\Omega))^3 \rightarrow H_{0, \Gamma, (\text{rot}; \Omega)}$ by setting $S_\varepsilon g : = S(\varepsilon g)$ it follows that $S_\varepsilon : H_{0, \Gamma, (\varepsilon, \text{div}; \Omega)}^0 \rightarrow H_{0, \Gamma, (\varepsilon, \text{div}; \Omega)}^0 \cap H_{0, \Gamma, (\text{rot}; \Omega)}$.

Then problem (3) can also be written as:

$$\begin{cases}
    \text{Find } U \in H_{0, \Gamma, (\varepsilon, \text{div}; \Omega)}^0 \cap H_{0, \Gamma, (\text{rot}; \Omega)} : \\
    U - (1 + \alpha^2) S_\varepsilon^* U = SM.
\end{cases}$$

Since the immersion $H_{0, \Gamma, (\text{rot}; \Omega)} \cap H_{0, \Gamma, (\varepsilon, \text{div}; \Omega)} ↪ (L^2(\Omega))^3$ is compact the Fredholm alternative holds for problem (7) in the Hilbert space $H_{0, \Gamma, (\varepsilon, \text{div}; \Omega)}^0$ and

$$R(I - (1 + \alpha^2) S_\varepsilon^*) = N(I - (1 + \alpha^2) S_\varepsilon^*)^\perp,$$

where $S_\varepsilon^*$ is the adjoint operator of $S_\varepsilon$ in $H_{0, \Gamma, (\varepsilon, \text{div}; \Omega)}$, and the symbol $\perp$ denotes the orthogonal space with respect to the scalar product of $H_{0, \Gamma, (\varepsilon, \text{div}; \Omega)}^0$ which is the $(L^2(\Omega))^3$-scalar product.

The proof can now be completed by following the last steps of the proof of Theorem 2.4 in [7]. We report it here for the sake of completeness.

Our target is to show that $SM \in N(I - (1 + \alpha^2) S_\varepsilon^*)^\perp$ if and only if $(M, \Phi) = 0$ for all solutions $\Phi$ of problem (5). We start by noticing that $S_\varepsilon^* = (\varepsilon S)^*$

$$(S_\varepsilon^* f, v) = (f, S_\varepsilon v) = (f, S(\varepsilon v)) = (S^* f, \varepsilon v) = (\varepsilon S^* f, v)$$

and that given $f \in (L^2(\Omega))^3$, $S^* f$ is coincident with the unique solution $TT f$ of the boundary value problem

$$\begin{cases}
    \text{Find } TT f \in H_{0, \Gamma, (\text{rot}; \Omega)} : \\
    ((\mu^{-1})^* \text{rot} TT f, \text{rot} v) + (\varepsilon TT f, v) = (f, v) \quad \forall v \in H_{0, \Gamma, (\text{rot}; \Omega)}.
\end{cases}$$

In fact

$$(S^* f, v) = \langle f, Sv \rangle = ((\mu^{-1})^* \text{rot} TT f, \text{rot} Sv) + (\varepsilon TT f, Sv) = (\text{rot} TT f, \mu^{-1} \text{rot} Sv) + (TT f, \varepsilon Sv) = (\mu^{-1} \text{rot} Sv, \text{rot} TT) + (\varepsilon Sv, TT) = (TT f, v) \quad \forall v \in H_{0, \Gamma, (\text{rot}; \Omega)}.$$

We will prove that $S^*[N(I - (1 + \alpha^2) S^*)]$ coincides with the set of solutions of problem (5). In fact if $f \in N(I - (1 + \alpha^2) S^*)$ then for all $v \in H_{0, \Gamma, (\text{rot}; \Omega)}$

$$(f, v) = (1 + \alpha^2)(\varepsilon S^* f, v)$$

but

$$(f, v) = ((\mu^{-1})^* \text{rot} S^* f, \text{rot} v) + (\varepsilon S^* f, v),$$

hence

$$(\mu^{-1})^* \text{rot} S^* f, \text{rot} v - \alpha^2(\varepsilon S^* f, v) = 0,$$
so $S^*[N(I - (1 + \alpha^2)S_\alpha^*\epsilon)]$ is contained in the set of solutions of problem (5). On the other hand if $\Phi \in H_{0,r_\alpha}(\text{rot}; \Omega)$ is a solution of problem (5), then for all $v \in H_{0,r_\alpha}(\text{rot}; \Omega)$

$((\mu^{-1})^*\text{rot}\Phi, \text{rot}v) + (\epsilon^*\Phi, v) = (1 + \alpha^2)(\epsilon^*\Phi, v).$

In other words $\Phi = S^*[1 + (1 + \alpha^2)\epsilon^*\Phi]$. Setting $f = (1 + \alpha^2)\epsilon^*\Phi$ we have $\Phi = S^*f$ and $f \in N(I - (1 + \alpha^2)S_\alpha^*)$ because

$(1 + \alpha^2)\epsilon^*S^*f = (1 + \alpha^2)\epsilon^*\Phi = f. \diamond$

Now, Theorem 2 is a consequence of the following result:

**Theorem 4.** Let the hypotheses reported in Section 2 be satisfied. Then both problems (4) with $M = 0$ and (5) have only the trivial solution.

**Proof.** First we see that if assumptions (H2) - (H6) are satisfied, any solution of both problems (4) with $M = 0$ and (5) is equal to zero a.e. in $\mathcal{A}_\epsilon \cup \mathcal{A}_m$. In fact let $u$ be a solution of problem (4) with $M = 0$, then

$$(\mu^{-1}\text{rot}u, \text{rot}u) - \alpha^2(\epsilon u, u) = 0.$$ 

In particular

$$\text{Im}[((\mu^{-1}\text{rot}u, \text{rot}u) - \alpha^2(\epsilon u, u)] = \text{Im}[[((\nu_s + i\nu_{ss})\text{rot}u, \text{rot}u) - \alpha^2((\epsilon_s - i\epsilon_{ss})u, u)] = (\nu_{ss}\text{rot}u, \text{rot}u) + \alpha^2(\epsilon_{ss}u, u) = 0. $$

From assumptions (H3)-(H6) it follows that $\|\text{rot}u\|_{L^2(\mathcal{A}_m)} = \|\text{rot}u\|_{L^2(\mathcal{A}_m)} = 0$. Hence $u$ is equal to zero a. e. in $\mathcal{A}_\epsilon$ and $\text{rot}u$ is equal to zero a. e. in $\mathcal{A}_m$. However $\alpha^2\epsilon u = \text{rot}(\mu^{-1}\text{rot}u)$ a. e. in $\Omega$. Hence, from assumption (H2) and since $\alpha \neq 0$ we have that $u$ is equal to zero a. e. in $\mathcal{A}_m$.

Analogously, let $\Phi$ be a solution of Problem 5. Then

$$(\nu_s - i\nu_{ss})\text{rot}\Phi, \text{rot}\Phi) - \alpha^2((\epsilon_s + i\epsilon_{ss})\Phi, \Phi) = 0.$$ 

In particular

$$\text{Im}[(\nu_s - i\nu_{ss})\text{rot}\Phi, \text{rot}\Phi) - \alpha^2((\epsilon_s + i\epsilon_{ss})\Phi, \Phi)] = -(\nu_{ss}\text{rot}\Phi, \text{rot}\Phi) - \alpha^2(\epsilon_{ss}\Phi, \Phi) = 0,$$

and we conclude as in the previous case.

Now the key points to conclude the proof are the techniques used by Hazard and Lenoir [22] to manage the discontinuity of the coefficients and the unique continuation principle for Maxwell equations that reads:

**Lemma 1.** Let $Q \subset \mathbb{R}^3$ be a bounded domain and $w \in H(\text{rot}; Q)$ be a solution of the following equation

$$\text{rot}(\mu^{-1}\text{rot}w) - \alpha^2\epsilon w = 0 \text{ in } Q.$$ 

Assume that $\epsilon$ and $\mu$ are hermitian symmetric matrices with entries in $C^2(Q)$ and uniformly positive definite in $Q$. If $w$ vanishes in an arbitrary small ball $B \subset Q$ then $w = 0$ in the whole domain $Q$.

For the proof of this lemma, see Leis [2]. There the coefficients $\epsilon$ and $\mu$ are assumed to be real symmetric but the proof is also valid for hermitian symmetric coefficients.

We will prove that problem (5) has only the trivial solution. The proof in the case of problem (4) with $M = 0$ is analogous.

We know that every solution $\Phi$ of (5) satisfies $\Phi|_{\mathcal{A}_\epsilon \cup \mathcal{A}_m} = 0$. In order to conclude that in fact $\Phi = 0$ in $\Omega$, we proceed as in [22] (where an analogous situation with $\epsilon$ and $\mu$ scalar real functions is considered). We set $\Omega_0 := \mathcal{A}_\epsilon \cup \mathcal{A}_m$ and we prove that if a solution of problem (5) vanishes in some subdomain $\Omega_j$, adjacent to $\Omega_k$ then it also vanishes in $\Omega_k$, $0 \leq j, k \leq L$. 

We consider the domain \( \tilde{\Omega}_k = U \cup \Omega_k \) where \( U \) is a ball centered at a point of \( \Gamma_{jk} := \partial \Omega_j \cap \partial \Omega_k \). Let \( \varepsilon_{mn} \) and \( \mu_{mn} \) denote respectively the restrictions to \( \tilde{\Omega}_k \) of \( \varepsilon_{kn} \) and \( \mathcal{M}_{mn}^k \) (introduced in assumption (H7)). We note that the following interface conditions

\[
(\Phi|_{|\Gamma_{jk}} \times n|_{|\Gamma_{jk}})|_{\Gamma_{jk}} = (\Phi|_{|\Gamma_{jk}} \times n|_{|\Gamma_{jk}})|_{\Gamma_{jk}} = 0
\]

are satisfied in a suitable weak sense, being \( n|_{|\Gamma_{jk}} \) a unit normal vector on \( \Gamma_{jk} \). It is clear that replacing \( \varepsilon \) and \( \mu \) by \( \tilde{\varepsilon} \) and \( \tilde{\mu} \) in \( \tilde{\Omega}_k \), the function \( \Phi \) satisfies equation (8) in \( \tilde{\Omega}_k \). But \( \Phi = 0 \) in \( U \cap \Omega_j \), hence, by Lemma 1, \( \Phi \) vanishes in \( \tilde{\Omega}_k \). 

\section{A compactness result}

\textbf{Lemma 2.} Let (H1) and the hypotheses on \( \Omega \) and \( \Gamma \) reported in Section 2 be satisfied, with both \( \Gamma_{\tau} \) and \( \Gamma_{\nu} \) non empty. Let, moreover, \( \varepsilon_n \) be uniformly positive definite in \( \Omega \). Then the immersion

\[
H_{0,\Gamma_{\tau}}(\text{rot}; \Omega) \cap H_{0,\Gamma_{\nu}}(\varepsilon, \text{div}; \Omega) \hookrightarrow (L^2(\Omega))^3
\]

is compact.

\textbf{Proof.} The proof is similar to the one of Lemma 3.1 in [7]. Some steps are actually identical but we include them here for the sake of completeness.

Let \( \{v_n\}_{n \in \mathbb{N}} \subset H_{0,\Gamma_{\tau}}(\text{rot}; \Omega) \cap H_{0,\Gamma_{\nu}}(\varepsilon, \text{div}; \Omega) \) be such that

\[
\|v_n\|_{0,\Omega} + \|\text{rot} v_n\|_{0,\Omega} + \|\varepsilon v_n\|_{0,\Omega} \leq 1 \quad \forall n \in \mathbb{N}. \tag{9}
\]

Let us introduce the complex function \( V_n \), which is the unique solution of the mixed boundary value problem

\[
V_n \in H^1_{0,\Gamma_{\tau}}(\Omega) : \quad b(V_n, \phi) := (\varepsilon \nabla V_n, \nabla \phi) = (\varepsilon v_n, \nabla \phi) \quad \forall \phi \in H^1_{0,\Gamma_{\tau}}(\Omega).
\]

Since the bilinear form \( b(\cdot, \cdot) \) is continuous and coercive in \( H^1_{0,\Gamma_{\tau}}(\Omega) \) there exists a constant \( K_0 > 0 \) such that

\[
\|V_n\|_{1,\Omega} \leq K_0 \quad \forall n \in \mathbb{N}. \tag{10}
\]

By applying Rellich theorem we can find a subsequence of \( \{V_n\}_n \) which is strongly convergent in \( L^2(\Omega) \) and since \( \|\varepsilon v_n\|_{0,\Omega} \leq 1 \), a subsequence of \( \{v_n\}_n \) such that \( \{\varepsilon v_n\}_n \) is weakly convergent in \( L^2(\Omega) \). Hence we have a subsequence \( \{n_k\}_k \) such that

\[
(\varepsilon (v_{n_k} - v_{n_l})), V_{n_k} - V_{n_l}) \underset{k, l \to \infty}{\to} 0
\]

But

\[
(\varepsilon (v_{n_k} - v_{n_l}), V_{n_k} - V_{n_l}) = (\varepsilon \nabla (V_{n_k} - V_{n_l}), \nabla (V_{n_k} - V_{n_l})) \to 0 \quad \text{as} \ k, l \to \infty.
\]

In particular

\[
\text{Re}[(\varepsilon \nabla (V_{n_k} - V_{n_l}), \nabla (V_{n_k} - V_{n_l}))] = (\varepsilon \nabla (V_{n_k} - V_{n_l}), \nabla (V_{n_k} - V_{n_l})) \to 0 \quad \text{as} \ k, l \to \infty,
\]

so, since \( \varepsilon_n \) is uniformly positive definite in \( \Omega \) we obtain that

\[
\{V_{n_k}\}_k \text{ is strongly convergent in } H^1(\Omega).
\]

Let us split now \( v_n := \nabla V_n + v_n^2 \). From (9) and (10) \( \|v_n^2\|_{0,\Omega} \leq 1 + K_0 \), \( \forall n \in \mathbb{N} \). Moreover \( \|\text{rot} v_n^2\|_{0,\Omega} = \|\text{rot} v_n\|_{0,\Omega} \leq 1 \), \( \forall n \in \mathbb{N} \), hence there exists a subsequence of \( \{v_n^2\}_n \) which is weakly convergent in \( H(\text{rot}; \Omega) \).
We note that $\varepsilon v_n^2 \in H_{0,\Gamma_v}^0(\text{div};\Omega)$. Defining
\[ H_{0,\Gamma_v,\text{flux};\Gamma_v,\Sigma}(\text{div};\Omega) := \{ v \in H_0^{0,\Gamma_v}(\text{div};\Omega) \mid (v, w) = 0 \forall w \in H_0^{0,\Gamma_v}(\text{rot};\Omega) \cap H_0^{0,\Gamma_v}(\text{div};\Omega) \} \]
(see [21] (4.13) and Proposition 6.3) we can decompose $\varepsilon v_n^2 = w_n + h_n$ with $w_n \in H_{0,\Gamma_v}^0(\text{rot};\Omega) \cap H_{0,\Gamma_v}^0(\text{div};\Omega)$ and $h_n \in H_{0,\Gamma_v,\text{flux};\Gamma_v,\Sigma}(\text{div};\Omega)$ and
\[ \|w_n\|_{0,\Omega} \leq \|\varepsilon v_n^2\|_{0,\Omega} \leq \|\varepsilon\|_{\infty,\Omega} (1 + K_0) =: K_1. \]
Since $H_{0,\Gamma_v}^0(\text{rot};\Omega) \cap H_{0,\Gamma_v}^0(\text{div};\Omega)$ is compactly embedded in $(L^2(\Omega))^3$ (see [21] Proposition 7.3), we deduce that there exists a subsequence of $\{w_n\}_n$ strongly convergent in $(L^2(\Omega))^3$.

Moreover we can choose $A_n \in [H_{0,\Gamma_v}^0(\text{rot};\Omega) \cap H_{0,\Gamma_v}^0(\text{div};\Omega)]^\perp$. As a matter of fact, since $A_n \in H_{0,\Gamma_v}^0(\text{div};\Omega)$, from [21] Proposition 6.3, $A_n$ can be decomposed as $A_n = A_n^1 + A_n^2$ with $A_n^1 \in H_{0,\Gamma_v}^0(\text{rot};\Omega) \cap H_{0,\Gamma_v}^0(\text{div};\Omega)$ and $A_n^2 \in [H_{0,\Gamma_v}^0(\text{rot};\Omega) \cap H_{0,\Gamma_v}^0(\text{div};\Omega)]^\perp$. It is clear that $A_n^2$ is also a solution of problem (11).

Since $A_n \in H_{0,\Gamma_v}^0(\text{rot};\Omega) \cap H_{0,\Gamma_v}^0(\text{div};\Omega) \cap [H_{0,\Gamma_v}^0(\text{rot};\Omega) \cap H_{0,\Gamma_v}^0(\text{div};\Omega)]^\perp$, from [21] Proposition 7.4 there exists a positive constant $c$ such that
\[ \|A_n\|_{0,\Omega} \leq c(\|\text{rot}A_n\|_{0,\Omega} + \|\text{div}A_n\|_{0,\Omega}) = c\|h_n\|_{0,\Omega}. \]
Moreover
\[ \|h_n\|_{0,\Omega} \leq \|\varepsilon v_n^2\|_{0,\Omega} \leq K_1, \]

\[ \|A_n\|_{0,\Omega} + \|\text{rot}A_n\|_{0,\Omega} \leq (1 + c)K_1. \]

Using again [21] Proposition 7.3 there exists a subsequence of $\{A_n\}_n$ strongly convergent in $(L^2(\Omega))^3$.

Hence we can select a subsequence $\{n_k\}_k$ such that
\[ \{v_{n_k}^2\}_k \text{ is weakly convergent in } H(\text{rot};\Omega) \]
\[ \{w_{n_k}\}_k \text{ is strongly convergent in } (L^2(\Omega))^3 \]
\[ \{A_{n_k}\}_k \text{ is strongly convergent in } (L^2(\Omega))^3 \]
therefore
\[ (v_{n_k}^2 - v_{n_l}^2, w_{n_k} - w_{n_l}) \to 0 \quad \text{as } k, l \to 0 \]
and
\[ (\text{rot}(v_{n_k}^2 - v_{n_l}^2), A_{n_k} - A_{n_l}) \to 0 \quad \text{as } k, l \to 0. \]

Since $\{v_{n_k}^2\}_n \subset H_{0,\Gamma_v}^0(\text{rot};\Omega)$ and $\{A_n\}_n \subset H_{0,\Gamma_v}^0(\text{rot};\Omega)$ using [21] Proposition 3.5
\[ (\text{rot}(v_{n_k}^2 - v_{n_l}^2), A_{n_k} - A_{n_l}) = (v_{n_k}^2 - v_{n_l}^2, \text{rot}(A_{n_k} - A_{n_l})) \]
\[ = (v_{n_k}^2 - v_{n_l}^2, h_{n_k} - h_{n_l}) \]
\[ = (v_{n_k}^2 - v_{n_l}^2, \varepsilon(v_{n_k}^2 - v_{n_l})) - (v_{n_k}^2 - v_{n_l}, h_{n_k} - h_{n_l}) \]

\[ \text{Re}(v_{n_k}^2 - v_{n_l}^2, \varepsilon(v_{n_k}^2 - v_{n_l})) \to 0 \quad \text{as } k, l \to 0, \]

In particular
\[ \text{Re}(v_{n_k}^2 - v_{n_l}^2, \varepsilon(v_{n_k}^2 - v_{n_l})) = (v_{n_k}^2 - v_{n_l}^2, \varepsilon(v_{n_k}^2 - v_{n_l})) \to 0 \quad \text{as } k, l \to 0, \]
and using that \( \varepsilon \) is uniformly positive definite we have
\[
\|v_{n_k}^2 - v_{n_l}^2\|^2 \longrightarrow 0 \quad \text{as } k, l \to 0,
\]
so
\[
\{v_{n_k}^2\} \text{ is strongly convergent in } (L^2(\Omega))^3.
\]
Recalling that
\[
v_n := \nabla V_n + v_{n_k}^2
\]
and that we can find a subsequence of \( \{V_n\} \) which is strongly convergent in \( H^1(\Omega) \) the lemma is proven. \( \diamond \)

**Remark 1.** An analogous result has been proved in [7] when \( \Gamma_\nu = \emptyset \). Moreover the proof in [7] can be easily modified to the case \( \Gamma_\tau = \emptyset \).

**Remark 2.** As a referee pointed out, if \( \varepsilon \) is a uniformly positive real function in \( L^\infty(\Omega) \) a result, in some respects more general, can be proved. As a matter of fact, F. Jochmann proved [28] that for \( 6/5 < q \leq 2 \), the space of complex valued functions \( u \in (L^2(\Omega))^3 \) with \( \text{rot } u \in (L^q(\Omega))^3 \) and \( \text{div}(\varepsilon u) \in L^q(\Omega) \) satisfying the mixed boundary conditions \( u \times n = 0 \) on \( \Gamma_\tau \), \( u \cdot n = 0 \) on \( \Gamma_\nu \), is compactly imbedded in \( (L^2(\Omega))^3 \).

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**References**


“Unique solvability for electromagnetic boundary value problems...”


